Part IB GEOMETRY (Lent 2019): Example Sheet 3

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1. Let V be the open subset $\{0 < u < \pi,\ 0 < v < 2\pi\}$, and $\sigma : V \to S^2$ be given by

$$\sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Prove that σ defines a smooth parametrization of a certain open subset of S^2. [*You may assume that functions such as* $\cos^{-1} : (-1, 1) \to (0, \pi)$ *and* $\tan^{-1} : (-\infty, \infty) \to (-\pi/2, \pi/2)$, $\cot^{-1} : (-\infty, \infty) \to (0, \pi)$ *are continuous.*]

2. Show that the tangent space to S^2 at a point $P = (x, y, z) \in S^2$ is the plane normal to the vector $\overrightarrow{OP}$, where O denotes the origin.

3. Show the stereographic projection map $\pi : S \setminus \{N\} \to \mathbb{C}$, where N denotes the north pole, defines a chart. Check that the spherical metric on $S \setminus \{N\}$ corresponds under π to the Riemannian metric on $\mathbb{C}$ given by $4(dx^2 + dy^2)/(1 + x^2 + y^2)^2$.

4. Let T denote the embedded torus in $\mathbb{R}^3$ obtained by revolving around the z-axis the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane. Using the formal definition of area in terms of a parametrization, calculate the surface area of T.

5. If one places S^2 inside a (vertical) circular cylinder of radius one, prove that the radial (horizontal) projection map from S^2 to the cylinder preserves areas (as first established by Archimedes). Deduce the existence of an atlas on S^2 for which the charts all preserve areas.

6. Using the geodesic equations, show directly that the geodesics in the hyperbolic plane are hyperbolic lines parametrized with constant hyperbolic speed. [Hint: In the upper half-plane model, prove that a geodesic curve between any two points of the positive imaginary axis L^+ is of the form claimed.]

7. For $a > 0$, let $S \subset \mathbb{R}^3$ be the circular half-cone defined by $z^2 = a(x^2 + y^2)$, $z > 0$, considered as an embedded surface. Show that S minus a ray through the origin is isometric to a suitable region in the plane with the usual length metric. [You should try to parametrize S minus a ray appropriately.] When is S minus the ray isometric to this region with the Euclidean metric induced from $\mathbb{R}^2$? When $a = 3$ show that no geodesic intersects itself, whereas when $a > 3$ show that there are geodesics (of infinite length) which do intersect themselves.

8. For a surface of revolution S, corresponding to an smooth curve $\eta : (a, b) \to \mathbb{R}^3$ given by $\eta(u) = (f(u), 0, g(u))$, where η' is never zero, η is a homeomorphism onto its image, and $f(u)$ is always positive, prove that the Gaussian curvature K is given by the formula

$$K = \frac{(f'g'' - f''g')g'}{f((f')^2 + (g')^2)^2}.$$

In the case when η is parametrized in such a way that $\|\eta'\| = 1$, prove that K is given by the formula $K = -f''/f$. Verify that the unit sphere has constant curvature 1. What about the sphere of radius R?

9. Using the results from the previous question, calculate the Gaussian curvature K for the hyperboloid of one sheet $x^2+y^2 = z^2+1$, and the hyperboloid of two sheets $x^2+y^2 = z^2-1$. Describe the qualitative properties of the curvature in these cases (sign and behaviour near infinity), and explain what you find using pictures of these surfaces.

For the embedded torus, as defined in Question 4, identify those points at which $K = 0$, $K > 0$ and $K < 0$. Verify the global Gauss–Bonnet theorem on the embedded torus.

10. Show that the embedded surface S with equation $x^2 + y^2 + c^2z^2 = 1$, where $c > 0$, is homeomorphic to the sphere. Deduce from the Gauss–Bonnet theorem that

$$\int_0^1 (1 + (c^2 - 1)u^2)^{-3/2} du = c^{-1}.$$

Can you find a direct verification of this formula?

11. Taking the disc model D for hyperbolic space, with Riemannian metric

$$\frac{4}{(1 - r^2)^2}(dr^2 + r^2 d\theta^2)$$

for plane polar coordinates (r, θ), let ρ be the hyperbolic distance $d(0, r)$. Express this Riemannian metric in terms of the cooordinates (ρ, θ). By considering geodesic polar coordinates around 0, what does this say for the curvature of D?

12. Given a Riemannian metric on the open subset U of $\mathbb{R}^2$ and a smooth curve $\gamma : [0, 1] \to U$, use the Cauchy - Schwarz inequality to show that $\ell(\gamma)^2 \leq En(\gamma)$ for ℓ length and En energy respectively. When do we have equality?

Now given points $P, Q \in U$, consider the set S of all smooth curves γ from P to Q. Deduce that if a smooth curve γ_0 minimises length over all $\gamma \in S$ and has constant speed then γ_0 minimises energy over all $\gamma \in S$ and thus is a geodesic.

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13. Show that Mercator's parametrization of the sphere (minus poles)

$$\sigma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

determines a chart (on the complement of a longitude) which sends latitudes and longitudes on the sphere to straight lines in the plane. Does σ preserve angles? What about area? (Is Greenland the same size as Africa?)

14. Show that the surface obtained by attaching 2 handles to a sphere (i.e. the surface of a 'doughnut with 2 holes') may be obtained topologically by suitably identifying the sides of a regular octagon. Indicate briefly how to extend your argument to show that a 'sphere with g handles' Σ_g may be obtained topologically by suitably identifying the sides of a regular $4g$-gon.

Show that Σ_g $(g > 1)$ may be given the structure of an abstract surface with a Riemannian metric, in such a way that it is locally isometric to the hyperbolic plane. [For this question, you will need the result from Q10 on Example Sheet 2.]

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.