**Part IB GEOMETRY (Lent 2019): Example Sheet 1**

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**1.** Let $f$ be an isometry from Euclidean $n$-space $\mathbb{R}^n$ onto Euclidean $m$-space $\mathbb{R}^m$. By following the proof in lectures of the classification of Euclidean isometries but for $n \neq m$ (or otherwise), show that in this case no such $f$ exists.

**2.** Suppose that $H$ is a hyperplane in Euclidean $n$-space $\mathbb{R}^n$ defined by $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant $c$. The reflection in $H$ is the map from $\mathbb{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{x} - c)\mathbf{u}$. Show that this is an isometry. Letting $P, Q$ be distinct points of $\mathbb{R}^n$, show that there is a reflection in some hyperplane that maps $P$ to $Q$. Show that the points fixed by this reflection are those which are equidistant from $P$ and $Q$.

**3.** Suppose that $l_1$ and $l_2$ are non-parallel lines in the Euclidean plane $\mathbb{R}^2$, and that $r_i$ denotes the reflection of $\mathbb{R}^2$ in the line $l_i$, for $i = 1, 2$. Show that the composite $r_1 r_2$ is a rotation of $\mathbb{R}^2$, and describe (in terms of the lines $l_1$ and $l_2$) the resulting fixed point and the angle of rotation.

**4.** Let $R(P, \theta)$ denote the clockwise rotation of $\mathbb{R}^2$ through an angle $\theta$ about a point $P$. If $A$, $B$, $C$ are the vertices, labelled clockwise, of a triangle in $\mathbb{R}^2$, prove that $R(A, \theta)R(B, \phi)R(C, \psi)$ is the identity if and only if $\theta = 2\alpha$, $\phi = 2\beta$ and $\psi = 2\gamma$, where $\alpha, \beta, \gamma$ denote the angles at, respectively, the vertices $A, B, C$ of the triangle $ABC$.

**5.** Prove that any isometry of the unit sphere $S^2$ is induced from (namely is the restriction of) an isometry of $\mathbb{R}^3$ which fixes the origin. Prove also that any matrix $A \in O(3, \mathbb{R})$ is the product of at most three reflections in planes through the origin. Deduce that an isometry of the unit sphere can be expressed as the product of at most three reflections in spherical lines. What isometries are obtained from the product of two reflections? What isometries are obtained from the product of three reflections?

**6.** By repeatedly applying the result from Question 2, when $P$ is either $\mathbf{0}$ or one of the standard basis vectors of $\mathbb{R}^n$, deduce that any isometry $T$ of $\mathbb{R}^n$ can be written as a composition of at most $n + 1$ reflections.

**7.** Suppose that $P$ is a point on the unit sphere $S^2$. For fixed $\rho$, with $0 < \rho < \pi$, the spherical circle with centre $P$ and radius $\rho$ is the set of points $Q \in S^2$ whose spherical distance from $P$ is $\rho$. Prove that a spherical circle of radius $\rho$ on $S^2$ has circumference $2\pi \sin \rho$ and area $2\pi(1 - \cos \rho)$.

**8.** Given a spherical line $l$ on the sphere $S^2$ and a point $P$ not on $l$, show that there is a spherical line $l'$ passing through $P$ and intersecting $l$ at right-angles. Prove that the minimum distance $d(P, Q)$ of $P$ from a point $Q$ on $l$ is attained at one of the two points of intersection of $l$ with $l'$, and that $l'$ is unique if this minimum distance is less than $\pi/2$.

**9.** Let $\pi : S^2 \to \mathbb{C}_\infty$ denote the stereographic projection map. Show that the spherical circles on $S^2$ biject under $\pi$ with the circles and straight lines on $\mathbb{C}$.

**10.** Show that any Möbius transformation $T \neq 1$ on $\mathbb{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of $S^2$ through a non-zero angle has exactly two fixed points $z_1$ and $z_2$, where $z_2 = -1/\bar{z}_1$. If now $T$ is a Möbius transformation with two fixed points $z_1$ and $z_2$

satisfying $z_2 = -1/\bar{z}_1$, prove that **either** $T$ corresponds to a rotation of $S^2$, **or** one of the fixed points, say $z_1$, is an *attractive* fixed point, i.e. for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

**11.** Prove that Möbius transformations of $\mathbb{C}_\infty$ preserve cross-ratios. If $u, v \in \mathbb{C}$ correspond to points $P, Q$ on $S^2$, and $d$ denotes the angular distance from $P$ to $Q$ on $S^2$, show that $-\tan^2 \frac{1}{2} d$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in an appropriate order.

**12.** Just as for geodesic triangulations, we can consider geodesic polygonal decompositions of the sphere $S^2$ or the unit square torus $T$ by convex geodesic polygons, where each polygon is the intersection of finitely many hemispheres (for the case of $S^2$), or is the bijective image of a convex Euclidean polygon in $\mathbb{R}^2$ under the map $\mathbb{R}^2 \to T$ (for the case of $T$). If the number of faces (polygons) is $F$, the number of edges is $E$ and the number of vertices is $V$, show that $F - E + V = 2$ for the sphere, and $= 0$ for the torus. We denote by $F_n$ the number of faces with precisely $n$ edges, and $V_m$ the number of vertices where precisely $m$ edges meet: show that $\sum_n nF_n = 2E = \sum_m mV_m$.

We suppose that each face has at least three edges, and at least three edges meet at each vertex. If $V_3 = 0$, deduce that $E \geq 2V$. If $F_3 = 0$, deduce that $E \geq 2F$. For the sphere, deduce that $V_3 + F_3 > 0$. For the torus, exhibit a polygonal decomposition with $V_3 = 0 = F_3$.

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**13.** Suppose we have some metric $d$ on the extended complex plane $\mathbb{C}_\infty$ which is invariant under the action of the Mobius group, that is for all $z_1, z_2 \in \mathbb{C}_\infty$ and all Möbius maps $f$ we have $d(f(z_1), f(z_2)) = d(z_1, z_2)$. What can we say about $d$? Deduce that (under the correspondence via stereographic projection), not all Möbius maps act on the sphere $S^2$ by rotations.

**14.** Given a geodesic polygonal decomposition of $S^2$ (as in Question 12 with the same notation) into spherical polygons, prove the identity

$$\sum_n (6-n)F_n = 12 + 2\sum_m (m-3)V_m.$$

If each face has at least three edges, and at least three edges meet at each vertex, deduce the inequality $3F_3 + 2F_4 + F_5 \geq 12$.

The surface of a football is decomposed into spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there? Demonstrate the existence of such a decomposition with each vertex contained in precisely one pentagon.

**15.** A spherical triangle $\triangle = ABC$ has vertices given by unit vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathbb{R}^3$, sides of length $a$, $b$, $c$, and angles $\alpha$, $\beta$, $\gamma$ (where the side opposite vertex $A$ is of length $a$ and the angle at $A$ is $\alpha$, etc.). The *polar* triangle $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{B}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$, and $\pi - a$, $\pi - b$, $\pi - c$ respectively. Deduce the formula

$$\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha \cos\beta.$$

**Note to the reader:** You should look at all the questions up to Question 12, and then any further questions you have time for.