Part IB GEOMETRY (Lent 2017): Example Sheet 2

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1. Let $U \subset \mathbb{R}^2$ be an open set equipped with a Riemannian metric $Edu^2 + 2Fdudv + Gdv^2$. For P any point of U, prove that there exists $\lambda > 0$ and an open neighbourhood V of P in U such that

$$(E - \lambda)du^2 + 2Fdudv + (G - \lambda)dv^2$$

is a Riemannian metric on V. [Hint: A real matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite iff $a > 0$ and $ac > b^2$.]

If U is path-connected, we define the distance between two points of U to be the infinum of the lengths of curves joining them; prove that this defines a metric on U. Give an example where this distance is not realized as the length of any curve joining them.

2. We define a Riemannian metric on the unit disc $D \subset \mathbb{C}$ by $(du^2 + dv^2)/(1 - (u^2 + v^2))$. Prove that the diameters (monotonically parametrized) are length minimizing curves for this metric. Defining the distance between two points of D as in Question 1, show that the distances in this metric are bounded, but that the areas are unbounded.

3. We let $V \subset \mathbb{R}^2$ denote the square given by $|u| < 1$ and $|v| < 1$, and define two Riemannian metrics on V given by

$$du^2/(1 - u^2)^2 + dv^2/(1 - v^2)^2, \quad \text{and} \quad du^2/(1 - v^2)^2 + dv^2/(1 - u^2)^2.$$

Prove that there is no isometry between the two spaces, but that an area-preserving diffeomorphism does exist.
[Hint: to prove that an isometry does not exist, show that in one space there are curves of finite length going out to the boundary, whilst in the other space no such curves exist.]

4. Let l denote the hyperbolic line in H given by a semicircle with centre $a \in \mathbb{R}$ and radius $r > 0$. Show that the reflection R_l is given by the formula

$$R_l(z) = a + \frac{r^2}{\bar{z} - a}.$$

5. If a is a point of the upper half-plane, show that the Möbius transformation g given by

$$g(z) = \frac{z - a}{z - \bar{a}}$$

defines an isometry from the upper half-plane model H to the disc model D of the hyperbolic plane, sending a to zero. Deduce that for points z_1, z_2 in the upper half-plane, the hyperbolic distance is given by $\rho(z_1, z_2) = 2\tanh^{-1}\big|(z_1 - z_2)/(z_1 - \overline{z_2})\big|$.

6. Suppose that z_1, z_2 are points in the upper half-plane, and suppose the hyperbolic line through z_1 and z_2 meets the real axis at points z_1^* and z_2^*, where z_1 lies on the hyperbolic line segment $z_1^* z_2$, and where one of z_1^* and z_2^* might be ∞. Show that the hyperbolic distance $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1^*, z_1, z_2, z_2^*, taken in an appropriate order.

7. Let C denote a hyperbolic circle of hyperbolic radius ρ in the upper half-plane model of the hyperbolic plane; show that C is also a Euclidean circle. If C has hyperbolic centre ic, find the radius and centre of C regarded as a Euclidean circle. Show that a hyperbolic circle of hyperbolic radius ρ has hyperbolic area

$$A = 2\pi(\cosh(\rho) - 1).$$

Describe how this function behaves for ρ large; compare the behaviour of the corresponding area functions in Euclidean and spherical geometry.

8. Given two points P and Q in the hyperbolic plane, show that the locus of points equidistant from P and Q is a hyperbolic line, the perpendicular bisector of the hyperbolic line segment from P to Q.

9. Show that any isometry g of the disc model D for the hyperbolic plane is **either** of the form (for some $a \in D$ and $0 \leq \theta < 2\pi$):

$$g(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

or of the form

$$g(z) = e^{i\theta}\frac{\bar{z}-a}{1-\bar{a}\bar{z}}.$$

10. Prove that a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$ has area

$$(n-2)\pi - \sum \alpha_i.$$

Show that for every $n \geq 3$ and every α with $0 < \alpha < (1 - \frac{2}{n})\pi$, there is a regular n-gon all of whose angles are α.

11. Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the perpendicular is unique. Given two ultraparallel hyperbolic lines, prove that the composite of the corresponding reflections has infinite order. [Hint: You may care to take the common perpendicular as a special line.]

12. Fix a point P on the boundary of D, the disc model of the hyperbolic plane. Give a description of the curves in D that are orthogonal to every hyperbolic line that passes through P.

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13. Let S^+ be the hyperboloid model of the hyperbolic plane. That is, consider the Lorenzian inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 - x_3y_3$ on $\mathbf{R}^3$, and let $S^+ = \{\mathbf{x} \in \mathbf{R}^3 : \langle \mathbf{x}, \mathbf{x} \rangle = -1,\ x_3 > 0\}$ with the Riemannian metric restricted from $\langle \mathbf{x}, \mathbf{y} \rangle$. Show that every plane P in $\mathbf{R}^3$ through 0 that meets S^+ can be written as $\{\mathbf{x} \in \mathbf{R}^3 : \langle \mathbf{x}, \mathbf{u} \rangle = 0\}$ for some vector $\mathbf{u} \in \mathbf{R}^3$ with $\langle \mathbf{u}, \mathbf{u} \rangle = 1$. Use this to write a formula for the reflection of S^+ in the hyperbolic line $S^+ \cap P$. Show that every hyperbolic line in S^+ arises this way.

14. Let l be a hyperbolic line and P a point on l. Show that there is a unique hyperbolic line l' through P making an angle α with l (in a given sense). If α, β are positive numbers with $\alpha + \beta < \pi$, show that there exists a hyperbolic triangle (one vertex at infinity) with angles 0, α and β. For any positive numbers α, β, γ, with $\alpha + \beta + \gamma < \pi$, show that there exists a hyperbolic triangle with these angles. [Hint: For the last part, you may need a continuity argument.]

15. For arbitrary points z, w in $\mathbb{C}$, prove the identity

$$|1 - \bar{z}w|^2 = |z - w|^2 + (1 - |z|^2)(1 - |w|^2).$$

Given points z, w in the unit disc model of the hyperbolic plane, prove the identity

$$\sinh^2(\frac{1}{2}\rho(z, w)) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

where ρ denotes the hyperbolic distance.

16. Let $\triangle$ be a hyperbolic triangle, with angles α, β, γ, and sides of length a, b, c (the side of length a being opposite the vertex with angle α, and similarly for b and c). Using the result from Question 15, and the Euclidean cosine rule, prove the hyperbolic cosine rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma.$$

[For a slicker proof of this result, and of the corresponding hyperbolic sine rule,

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma},$$

via the hyperboloid model of hyperbolic space, consult P.M.H. Wilson, Curved Spaces, § 5.7.]

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.