IB GEOMETRY LENT 2014

EXAMPLE SHEET 3

Notation: on this sheet, $\mathbb{H}$ denotes the hyperbolic plane, and is used in statements that make sense in any model. H is the upper half-plane model of $\mathbb{H}$, and D the disk model.

1. Find the perimeter and area of a circle of radius r on S^2; of a circle of radius r in the hyperbolic plane.

2. If L is the hyperbolic line in H given by a Euclidean semicircle with center $a \in \mathbb{R}$ and radius $r > 0$, show that reflection in the line L is given by the formula
$$\rho_L(z) = a + \frac{r^2}{\overline{z} - a}.$$

3. If w is a point in the upper half-plane, show that the Mobius transformation φ given by $\varphi(z) = (z - w)/(z - \overline{w})$ defines an isometry from H to the disk model D of the hyperbolic plane. Deduce that if $z, w \in H$, the hyperbolic distance from z to w is given by $d(z, w) = 2\tanh^{-1} |(z - w)/(z - \overline{w})|$.

4. Let C be a hyperbolic circle in H; show that C is also a Euclidean circle. If C has hyperbolic center ic ($c \in \mathbb{R}^+$) and hyperbolic radius r, find the radius and center of C regarded as a Euclidean circle.

5. Prove that the area of a convex hyperbolic n-gon with interior angles $\alpha_1, \dots, \alpha_n$ is $(n - 2)\pi - \sum \alpha_i$. Show that for every $n \geq 3$ and every α with $0 \leq \alpha \leq (1 - \frac{2}{n})\pi$, there is a regular hyperbolic n-gon all of whose interior angles are α.

6. Let L be a hyperbolic line, and let $\mathbf{p} \in \mathbb{H}$ be a point not on L. Show there is a unique hyperbolic line passing through $\mathbf{p}$ and perpendicular to L. If L is a spherical line and $\mathbf{p} \in S^2$ is a point not on L, show that there is a spherical line passing through $\mathbf{p}$ and perpendicular to L, but this line may not be unique.

7. Show that two hyperbolic lines L_1, L_2 have a common perpendicular if and only if they are ultraparallel, and that in this case the perpendicular is unique. Let $\rho_i : \mathbb{H} \to \mathbb{H}$ be the reflection in L_i. Show that if L_1 and L_2 are ultraparallel, $\rho_1 \circ \rho_2$ has infinite order. (*Hint:* take the common perpendicular as a special line.)

8. Show that two distinct Euclidean circles intersect in at most two points; deduce that the same holds for hyperbolic circles. If A_1, A_2, A_3 and B_1, B_2, B_3 are two sets of non-colinear points in $\mathbb{H}$, and $d(A_i, A_j) = d(B_i, B_j)$ for all choices of i and j, deduce that there is a unique $\varphi \in \mathrm{Isom}(\mathbb{H})$ with $\varphi(A_i) = B_i$.

9. Show that there is a constant k such that no hyperbolic triangle contains a hyperbolic circle of radius greater than k. What is the smallest such value of k? Deduce that if $\triangle ABC$ is a hyperbolic triangle, then any point on $\overline{BC}$ is within hyperbolic distance $2k$ of either $\overline{AB}$ or $\overline{AC}$.

10. * Fix a point $\mathbf{p}$ on the boundary of D, and let L be a hyperbolic line through $\mathbf{p}$. Viewing L as a Euclidean circle, show that the center of L lies on the (Euclidean) line tangent to ∂D at $\mathbf{p}$. Let $\mathbf{q}$ be a point in D not on L, and let L_1 and L_2 be the two horoparallels to L passing through $\mathbf{q}$. Express the angle between L_1 and L_2 in terms of the hyperbolic distance from $\mathbf{q}$ to L.

11. Suppose we have a polygonal decomposition of S^2 by convex geodesic polygons, where each polygon is contained in some hemisphere. Denote by F_n the number of faces with precisely n edges, and V_m the number of vertices where precisely m edges meet; show that $\sum_n nF_n = 2E = \sum_m mV_m$.

    Suppose that $V_i = F_i = 0$ for $i < 3$. If in addition $V_3 = 0$, deduce that $E \geq 2V$. Similarly, if $F_3 = 0$, deduce that $E \geq 2F$. Conclude that $V_3 + F_3 > 0$. Prove the identity
$$\sum_n (6-n)F_n = 12 + 2\sum_m (m-3)V_m.$$
    Deduce that $3F_3 + 2F_4 + F_5 \geq 12$. The surface of a football is decomposed into spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there?

12. Let T be the torus obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z-axis. Find the Gauss curvature K of T, and identify the points on T where K is positive, negative, and zero. Verify that
$$\int_T K \; dA = 0.$$

13. Show that the embedded surface given by the equation $x^2 + y^2 + c^2z^2 = 1$ $(c > 0)$ is homeomorphic to S^2. Deduce from the global Gauss-Bonnet theorem that
$$\int_0^1 (1 + (c^2-1)u^2)^{-3/2} \; du = c^{-1}.$$

14. * Show that a genus two surface can be obtained by appropriately identifying the sides of a regular octogon. Use problem 5 to show that the genus two surface admits a Riemannian metric with constant curvature $K = -1$.

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