IB GEOMETRY **LENT 2014**

EXAMPLE SHEET 2

1. Let $g = f(x)\,dx^2$ be an arbitrary Riemannian metric on $\mathbb{R}$. Show that $(\mathbb{R}, g)$ is isometric to the Euclidean metric $g_E = dx^2$ on an open interval $(a, b) \subset \mathbb{R}$. (Hint: parametrize by arc-length.)

2. Let S be the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1\}$. Check directly that the curve $\gamma(t) = (\cos at, \sin at, bt + c)$ satisfies the relation $\gamma''(t) \perp S$.

3. Let S be the surface obtained by rotating the curve $x = f(z)$ around the z-axis. Show that for any fixed value of θ, the curve $\gamma(t) = (f(t)\cos\theta, f(t)\sin\theta, t)$ is a geodesic on S. For which values of z is the curve $\gamma(t) = (f(z)\cos t, f(z)\sin t, z)$ a geodesic?

4. For $a > 0$, let S be the circular half-cone $\{(x, y, z) \mid z^2 = a(x^2 + y^2), z > 0\}$. Show that S minus a ray through the origin is locally isometric to the Euclidean plane. (Hint: identify the edges of a circular sector.) When $a = 3$, give an explicit formula for the geodesics on S and show that no geodesic intersects itself. For $a > 3$, show that there are geodesics which intersect themselves.

5. Let $\mathbf{x}$ and $\mathbf{y}$ be points in $\mathbb{R}^n$. Show that the line segment from $\mathbf{x}$ to $\mathbf{y}$ is the unique minimizer of the energy functional $E_g : \mathcal{P}_{\mathbf{x},\mathbf{y}}(\mathbb{R}^n) \to \mathbb{R}$.

6. Let V be the set of smooth functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(t)\,dt = k$. If $F : V \to \mathbb{R}$ is given by $F(f) = \int_0^1 f(t)^2\,dt$, show that the only critical point of F is the constant function $f(t) = k$. Deduce that geodesics have constant speed.

7. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\sigma(u, v) = (u, v, F(u, v))$ Show that the Gauss curvature of σ is given by

$$\frac{F_{uu}F_{vv} - F_{uv}^2}{(1 + F_u^2 + F_v^2)^2}.$$

8. Let γ be an embedded curve in the xz plane given by the parametrization $\gamma(t) = (f(t), 0, g(t))$, and let S be the surface obtained by rotating γ around the z-axis. Show that the Gauss curvature of S is

$$K = \frac{g'(f'g'' - f''g')}{f(f'^2 + g'^2)^2}.$$

If γ is parametrized so as to have unit speed, show this reduces to $K = -f''/f$. How is the sign of K related to the concavity of γ?

9. Using the previous question, compute the Gauss curvature of the surfaces given by the equations $x^2 + y^2 - z^2 = 1$ and $x^2 + y^2 - z^2 = -1$. Describe the qualitative properties of the curvature in these cases (sign and behavior near ∞) and explain what you find using pictures of these surfaces.

10. Let S be a compact embedded surface in $\mathbb{R}^3$. By considering the smallest closed ball centered at the origin which contains S, show that the Gauss curvature must be strictly positive at some point of S.

11. Let D be an open disc centered at the origin in $\mathbb{R}^2$. Give D a Riemannian metric of the form $(dx^2+dy^2)/f(r)^2$, where $r = \sqrt{x^2 + y^2}$ and $f(r) > 0$. Show that the Gauss curvature of this metric is $K = ff'' - (f')^2 + ff'/r$.

12. Let U be an open set in $\mathbb{R}^2$ equipped with a Riemannian metric, and let $\mathbf{p}$ be a point in U. Suppose that in geodesic polar coordinates centered at $\mathbf{p}$, the metric has the form $g = dr^2 + h(r)^2 d\theta^2$, and that the Taylor series of h converges to h in some neighborhood of $r = 0$. Show that the first two nontrivial terms of the Taylor series have the form $h(r) = r - Kr^3/6 + \ldots$, where K is the Gauss curvature at $\mathbf{p}$. Deduce that if $A(r)$ is the area of the disc of radius r centered at $\mathbf{p}$, then $A(r) = \pi(r^2 - Kr^4/12 + \ldots)$.

13. * Suppose S is a surface of revolution obtained by rotating a curve γ in the xz-plane about the z-axis. Find conditions on γ so that the Gauss curvature of S is identically -1. (Hint: parametrize by arc length.) Can you find γ so that the surface S is geodesically complete?

14. * Let S be an embedded surface in $\mathbb{R}^3$, let $\mathbf{x}$ be a point on S, and let $\mathbf{n}$ be the unit normal to S at $\mathbf{x}$. For $\mathbf{v} \in T_{\mathbf{x}}S$, let $\gamma_{\mathbf{v}}$ be the plane curve obtained as the intersection of S with the plane passing through $\mathbf{x}$ and spanned by the vectors $\mathbf{n}$ and $\mathbf{v}$. We can view the second fundamental form of S as a bilinear map $B_{II} : T_{\mathbf{x}}S \times T_{\mathbf{x}}S \to \mathbb{R}$. Show that if $\mathbf{v} \in T_{\mathbf{x}}S$ has $|\mathbf{v}| = 1$, then $B_{II}(\mathbf{v}, \mathbf{v})$ is the curvature of $\gamma_{\mathbf{v}}$ at the point $\mathbf{x}$.

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