IB GEOMETRY **LENT 2014**

EXAMPLE SHEET 1

1. If R_1 and R_2 are two rays in $\mathbb{R}^n$, show that there is some $\phi \in \text{Isom}(\mathbb{R}^n)$ with $\phi(R_1) = R_2$. Describe the set of all such ϕ.

2. Suppose that L_1 and L_2 are non-parallel lines in $\mathbb{R}^2$, and that $\rho_i : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the reflection in the line L_i for $i = 1, 2$. Show that the composition $\rho_1\rho_2$ is a rotation. Describe the center and angle of rotation in terms of L_1 and L_2.

3. Suppose that H is a hyperplane in $\mathbb{R}^n$ defined by the equation $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant c. The reflection in H is the map $\mathbb{R}^n \to \mathbb{R}^n$ given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u} - c)\mathbf{u}$. Show this is an isometry. If $\mathbf{x}$ and $\mathbf{y}$ are points of $\mathbb{R}^n$, show that there is a hyperplane $H_{\mathbf{x},\mathbf{y}}$ so that reflection in $H_{\mathbf{x},\mathbf{y}}$ maps $\mathbf{x}$ to $\mathbf{y}$.

4. Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct points in $\mathbb{R}^n$. Show that if $|\mathbf{x}-\mathbf{z}| = |\mathbf{y}-\mathbf{z}|$, then $\mathbf{z} \in H_{\mathbf{x},\mathbf{y}}$. Deduce that every isometry of $\mathbb{R}^n$ is the product of at most $n+1$ reflections.

5. Suppose that $\phi \in \text{Isom}(\mathbb{R}^2)$. Show that there is either a point $\mathbf{x} \in \mathbb{R}^2$ with $\phi(\mathbf{x}) = \mathbf{x}$ or a line L with $\phi(L) = L$. Conclude that ϕ is either (a) a translation, (b) a rotation, (c) a reflection, or (d) a composition $\rho \circ T$, where ρ is reflection in a line L and and T is translation by some vector parallel to L. How many reflections are needed to generate an isometry of each type?

6. Let G be a finite subgroup of $\text{Isom}(\mathbb{R}^n)$. By considering the barycentre (*i.e.* average) of the orbit of the origin under G, show that G fixes some point of $\mathbb{R}^n$. If $n = 2$, show that G is either cyclic or *dihedral* (that is $D_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and for $n \geq 3$, D_{2n} is the full symmetry group of a regular $2n$-gon.)

7. Suppose $\gamma : [a, b] \to \mathbb{R}^2$ is a smooth curve parametrized by arc length. Let $\mathbf{n}(s)$ be the unit normal vector to $\gamma'(s)$, chosen so that $(\gamma'(s), \mathbf{n}(s))$ is a positively oriented basis of $\mathbb{R}^2$. Show that $\gamma''(s) = \kappa(s)\mathbf{n}(s)$ for some $\kappa(s) : [a, b] \to \mathbb{R}$. $\kappa(s)$ is the *curvature* of γ at $\gamma(s)$. Show that the curvature of a circle of radius R is $1/R$.

   *If $\gamma : [a, b] \to \mathbb{R}^2$ is an arbitrary smooth curve in $\mathbb{R}^2$, we define its curvature at $\gamma(t)$ to be the curvature of γ's reparametrization by arc length. Suppose $\gamma : [0, 2\pi] \to \mathbb{R}^2$ is a smooth simple closed curve given in polar coordinates (r, θ) by

   $$r = r(t) > 0, \quad \theta = t, \quad \text{where} \quad r(0) = r(2\pi), \quad r'(0) = r'(2\pi).$$

   Show that the *total curvature* $\int_\gamma \kappa(s)\, ds = 2\pi$. Relate this to the theorem in plane geometry that says that the sum of the exterior angles of a convex polygon is 2π.

   Can you find a smooth closed curve γ whose total curvature is 0?

8. Let $\sigma : U \to \mathbb{R}^3$ be an embedded parametrized surface, and let $\mathbf{n}_{(u,v)}$ be the unit normal to σ at $\sigma(u,v)$. Let C be a compact subset of U. Define a map $\Sigma : C \times \mathbb{R} \to \mathbb{R}^3$ by $\Sigma(u,v,w) = \sigma(u,v) + w\mathbf{n}_{(u,v)}$, and let $V(t)$ be the volume of $\Sigma(C \times [0,t]) \subset \mathbb{R}^3$.

   (a) Assuming that for small ϵ, Σ is injective when restricted to $C \times [0,\epsilon]$, show that $V'(0)$ is the surface area of $\sigma(C)$. (This says we can find the surface area by covering $\sigma(C)$ with a thin layer of paint and seeing how much paint we used.)

   (b) * Prove that the assumption holds.

9. For each map $\sigma : U \to \mathbb{R}^3$, find the Riemannian metric on U induced by σ. Sketch the image of σ in $\mathbb{R}^3$.

   (a) $U = \{(u,v) \in \mathbb{R}^2 \mid u > v\}$, $\sigma(u,v) = (u+v, 2uv, u^2+v^2)$.

   (b) $U = \{(r,z) \in \mathbb{R}^2 \mid r > 0\}$, $\sigma(r,z) = (r\cos z, r\sin z, z)$.

   (c) $U = (0,2\pi) \times (0,2\pi)$, $\sigma(\theta,\phi) = ((a+b\cos\phi)\cos\theta, (a+b\cos\phi)\sin\theta, b\sin\phi)$ where $a > b$.

10. Let S be the complement of the points $(0,0,\pm 1)$ in S^2, and let $C = \{(x,y,z) \mid x^2 + y^2 = 1\}$ be a cylinder of radius 1. If $\phi : S \to C$ is the map given by radial projection from the z axis, show that ϕ is area-preserving.

11. Define a Riemannian metric on the unit disk $D \subset \mathbb{R}^2$ by $(du^2 + dv^2)/(1 - u^2 - v^2)$. Prove that the diameters are length-minimizing curves for this metric. Show that distances in this metric are bounded, but areas can be unbounded.

12. Let $V \subset \mathbb{R}^2$ be the square $|u|, |v| < 1$, and define two Riemannian metrics on V by

$$\frac{du^2}{(1-u^2)^2} + \frac{dv^2}{(1-v^2)^2} \quad \text{and} \quad \frac{du^2}{(1-v^2)^2} + \frac{dv^2}{(1-u^2)^2}.$$

   Prove that there is no isometry between the spaces, but that there is an area preserving diffeomorphism between them. (*Hint:* show that in one space there are curves of finite length going out to the boundary, while in the other no such curves exist.)

13. Suppose U is an open subset of $\mathbb{C}$, and that $f : U \to \mathbb{C}$ is a holomorphic map. (If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + iy$, this means that $D_w f(iz) = iD_w(z)$). If we equip the range of f with the Euclidean metric $dx^2 + dy^2$, what is the Riemannian metric on U induced by f? Deduce that if $f'(z) \neq 0$ for $z \in U$, then f is conformal.

14. * By evaluating the integral $\int_{\mathbb{R}^{n+1}} e^{-|\mathbf{x}|^2}\, d\mathbf{x}$ in two ways, express the n-dimensional volume of the n-dimensional sphere S^n in terms of the function $\Gamma(k) = \int_0^\infty r^{k-1}e^{-r}dr$. Show that $\Gamma(k+1) = k\Gamma(k)$, and thus compute the volume of S^n.

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