Part IB GEOMETRY, Examples sheet 3 (Lent 2011, Burt Totaro)

(1) Show that the tangent space to S^2 at a point $P = (x, y, z) \in S^2$ is the plane normal to the vector $\overrightarrow{OP}$, where O denotes the origin.

(2) Let V be the open subset $\{0 < u < \pi,\ 0 < v < 2\pi\}$ of the plane. Define $\sigma : V \to S^2$ by

$$\sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Prove that σ defines a smooth parametrization of a certain open subset of S^2. [*You may use that* $\cos^{-1}$ *is continuous on* $[-1, 1]$, *and* $\tan^{-1}, \cot^{-1}$ *are continuous on* $(-\infty, \infty)$.]

(3) Show that the stereographic projection map $\pi : S - \{N\} \to \mathbf{C}$, where N denotes the north pole, defines a chart. Check that the spherical metric on $S - \{N\}$ corresponds under π to the Riemannian metric on $\mathbf{C}$ given by $4(dx^2 + dy^2)/(1 + x^2 + y^2)^2$.

(4) Let T denote the embedded torus in $\mathbf{R}^3$ obtained by revolving around the z-axis the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane. Using the formal definition of area in terms of a parametrization, calculate the surface area of T.

(5) If one places S^2 inside a (vertical) circular cylinder of radius one, prove that the radial (horizontal) projection map from S^2 to the cylinder preserves areas (this is usually known as *Archimedes's theorem*). Deduce the existence of an atlas on S^2 for which the charts all preserve areas and the transition functions have derivatives with determinant one.

(6) Using the geodesic equations, show directly that the geodesics in the hyperbolic plane are hyperbolic lines parametrized with constant speed. [Hint: In the upper half-plane model, prove that a geodesic curve between two points on the positive imaginary axis L^+ is of the form claimed.]

(7) For $a > 0$, let $S \subset \mathbf{R}^3$ be the circular half-cone defined by $z^2 = a(x^2 + y^2)$, $z > 0$, considered as an embedded surface. Show that S minus a ray through the origin is isometric to a suitable region in the plane. [Intuitively: you can glue a piece of paper to form a cone, without any crumpling of the paper.] When $a = 3$, give an explicit formula for the geodesics on S and show that no geodesic intersects itself. When $a > 3$, show that there are geodesics (of infinite length) which intersect themselves.

(8) Let $S \subset \mathbf{R}^3$ denote the graph of a smooth function F (defined on some open subset of $\mathbf{R}^2$), $z = F(x, y)$. Show that S is a smooth embedded surface, and that its curvature at a point $(x, y, z) \in S$ is the value taken at (x, y) by

$$(F_{xx}F_{yy} - F_{xy}^2)/(1 + F_x^2 + F_y^2)^2.$$

(9) For a surface of revolution S, corresponding to an embedded curve $\eta : (a, b) \to \mathbf{R}^3$ with $\eta(u) = (f(u), 0, g(u))$, where η' is never zero, η is a homeomorphism onto its image, and $f(u)$ always positive, prove that the Gaussian curvature K is given by the formula

$$K = \frac{(f'g'' - f''g')g'}{f((f')^2 + (g')^2)^2}.$$

In the case where η is parametrized in such a way that $\|\eta'\| = 1$, prove that K is given by the formula $F = -f''/f$. Verify that the unit sphere has constant curvature 1.

(10) Using the results of the previous question, calculate the Gaussian curvature K for the hyperboloid of one sheet $x^2 + y^2 = z^2 + 1$ and the hyperboloid of two sheets $x^2 + y^2 = z^2 - 1$. Describe the qualitative properties of the curvature in these cases (sign and behavior near infinity), and explain what you find using pictures of these surfaces.

For the embedded torus, as defined in question 4, identify those points at which the curvature is negative, zero, or positive. Verify the global Gauss-Bonnet theorem on the embedded torus.

(11) For T the locally Euclidean torus, consider two charts obtained by projecting two different translates of the open unit square from $\mathbf{R}^2$ into T. Show that the corresponding transition function is not in general a translation, although it is locally a translation. What is the minimum number of such charts needed to form an atlas?

(12) Suppose we have a Riemannian metric of the form $|dz|^2/h(r)^2$ on some open disc $D(0, \delta)$ centered at the origin in $\mathbf{C}$ (possibly all of $\mathbf{C}$), where $h(r) > 0$ for all $r < \delta$. Show that the curvature K of this metric is given by the formula $K = hh'' - (h')^2 + hh'/r$.

(13) Show that the embedded surface $x^2+y^2+c^2z^2 = 1$, where $c > 0$, is homeomorphic to the sphere. Deduce from the Gauss-Bonnet theorem that

$$\int_0^1 (1 + (c^2 - 1)u^2)^{-3/2} du = c^{-1}.$$

Can you find a direct verification of this formula?

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(14) Given a smooth curve $\Gamma : [0, 1] \to S$ on an abstract surface S with a Riemannian metric, show that the length l is unchanged under reparametrizations of the form $f : [0, 1] \to [0, 1]$ with $f'(t) > 0$ for all $t \in [0, 1]$. Prove that if the curve Γ is an immersion, meaning that $\Gamma'(t) \neq 0$ for all t, then Γ can be reparametrized to a curve with constant speed.

(15) Let S be an embedded surface in $\mathbf{R}^3$ which is closed and bounded. By considering the smallest closed ball centred on the origin which contains S, or otherwise, show that the Gaussian curvature must be strictly positive at some point of S. Deduce that the locally Euclidean metric on the torus T cannot be realized as the first fundamental form of a smooth embedding of T in $\mathbf{R}^3$.

(16) Show that a 2-holed torus may be obtained topologically by suitably identifying the sides of a regular octagon. Indicate briefly how to extend your argument to show that a g-holed torus may be obtained topologically by suitably identifying the sides of a regular $4g$-gon.

Show that a g-holed torus ($g > 1$) may be given the structure of an abstract surface with a Riemannian metric which is locally isometric to the hyperbolic plane. [Use question 10 on Examples Sheet 2.]

Note to the reader: You should look at all the questions up to Question 13, and then any further questions you have time for.