Part IB Geometry, Examples sheet 1 (Lent 2010)

Burt Totaro

(1) Suppose that H is a hyperplane in Euclidean n-space $\mathbf{R}^n$ defined by $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant c. The reflection in H is the map from $\mathbf{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u} - c)\mathbf{u}$. Show that this is an isometry. Letting P, Q be points of $\mathbf{R}^n$, show that there is a reflection in some hyperplane that maps P to Q.

(2) Suppose that l_1 and l_2 are non-parallel lines in the Euclidean plane $\mathbf{R}^2$, and that r_i denotes the reflection of $\mathbf{R}^2$ in the line l_i, for $i = 1, 2$. Show that the composite $r_1 r_2$ is a rotation of $\mathbf{R}^2$, and describe (in terms of the lines l_1 and l_2) the resulting fixed point and angle of rotation.

(3) Let $R(P, \theta)$ denote the clockwise rotation of $\mathbf{R}^2$ through an angle θ about a point P. If A, B, C are the vertices, labelled clockwise, of a triangle in $\mathbf{R}^2$, prove that $R(A, \theta)R(B, \phi)R(C, \psi)$ is the identity if and only if $\theta = 2\alpha$, $\phi = 2\beta$, and $\psi = 2\gamma$, where α, β, γ denote the angles at the vertices A, B, C of the triangle ABC.

(4) Show from first principles that a (continuous) curve of shortest length between two points in Euclidean space is a straight line segment, parametrized monotonically.

(5) Let G be a finite subgroup of $\mathrm{Isom}(\mathbf{R}^m)$. By considering the barycentre (i.e., average) of the orbit of the origin under G, or otherwise, show that G fixes some point of $\mathbf{R}^m$. If G is a finite subgroup of $\mathrm{Isom}(\mathbf{R}^2)$, show that it is either cyclic or *dihedral* (that is, $D_4 = \mathbf{Z}/2 \times \mathbf{Z}/2$, or, for $n \geq 3$, the full symmetry group D_{2n} of a regular n-gon).

(6) Prove that any isometry of the unit sphere is induced from an isometry of $\mathbf{R}^3$ which fixes the origin. Prove that any matrix $A \in O(3, \mathbf{R})$ is the product of at most three reflections in planes through the origin. Deduce that an isometry of the unit sphere can be expressed as the product of at most three reflections in spherical lines. What isometries are obtained as the product of two reflections? What isometries can be written as the product of three reflections and no fewer?

(7) Let P be a point on the unit sphere S^2. For fixed ρ, with $0 < \rho < \pi$, the *spherical circle* with centre P and radius ρ is the set of points $Q \in S^2$ whose spherical distance from P is ρ. Prove that a spherical circle of radius ρ on S^2 has circumference $2\pi \sin \rho$ and area $2\pi(1 - \cos \rho)$.

(8) Given a spherical line l on the sphere S^2 and a point P not on l, show that there is a spherical line l' passing through P and intersecting l at right angles. Prove that the minimum distance $d(P, Q)$ from P to a point Q on l is attained at one of the two points of intersection of l with l', and that l' is unique if this minimum distance is less than $\pi/2$.

(9) Let $\pi : S^2 \to \mathbf{C}_\infty$ denote the stereographic projection map. Show that π gives a bijection between the spherical circles on S^2 and the circles and straight lines on $\mathbf{C}$.

(10) Show that any Möbius transformation $T \neq 1$ on $\mathbf{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of S^2 through a nonzero angle has exactly two fixed points z_1 and z_2, where $z_2 = -1/\overline{z_1}$. If now T is a Möbius transformation with two fixed points z_1

and z_2 satisfying $z_2 = -1/\overline{z_1}$, prove that *either* T corresponds to a rotation of S^2, *or* one of the fixed points, say z_1, is an *attractive* fixed point, i.e., for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

(11) Prove that Möbius transformations of $\mathbf{C}_\infty$ preserve cross-ratios. If $u, v \in C$ correspond to points P, Q on S^2, and d denotes the angular distance from P to Q on S^2, show that $-\tan^2(d/2)$ is the cross ratio of the points $u, v, -1/\overline{u}, -1/\overline{v}$, taken in an appropriate order (which you should specify).

(12) Suppose we have a polygonal decomposition of the sphere S^2 or the locally Euclidean torus T by convex geodesic polygons, where each polygon is contained in some hemisphere (for the case of S^2), or is the bijective image of a Euclidean polygon in $\mathbf{R}^2$ under the covering map $\mathbf{R}^2 \to T$ (for the case of T). If the number of vertices is V, the number of edges is E, and the number of faces (polygons) is F, show that $V - E + F$ equals 2 for the sphere, and 0 for the torus. We denote by F_n the number of faces with precisely n edges, and V_m the number of vertices where precisely m edges meet; show that $\sum_n nF_n = 2E = \sum_m mV_m$.

We suppose that each face has at least three edges, and that at least three edges meet at each vertex. If $V_3 = 0$, deduce that $E \geq 2V$. If $F_3 = 0$, deduce that $E \geq 2F$. For the sphere, deduce that $V_3 + F_3 > 0$. For the torus, exhibit a polygonal decomposition such that $V_3 = 0 = F_3$.

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(13) For every spherical triangle $\triangle = ABC$, show that $a < b + c$, $b < c + a$, $c < a + b$, and $a + b + c < 2\pi$. Conversely, show that for any three positive numbers a, b, c less than π satisfying the above conditions, we have $\cos(b + c) < \cos a < \cos(b - c)$, and that there is a spherical triangle (unique up to isometries of S^2) with those sides.

(14) A spherical triangle $\triangle = ABC$ has vertices given by unit vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbf{R}^3$, sides of length a, b, c, and angles α, β, γ (where the side opposite vertex A is of length a and the angle at A is α, and so on). The *polar* triangle $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$, and $\mathbf{A} \times \mathbf{B}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha, \pi - \beta, \pi - \gamma$ and $\pi - a, \pi - b, \pi - c$ respectively. Deduce the formula

$$\sin\alpha \sin\beta \cos c = \cos\gamma + \cos\alpha\cos\beta.$$

(15) Two spherical triangles $\triangle_1, \triangle_2$ on a sphere S^2 are said to be *congruent* if there is an isometry of S^2 that takes $\triangle_1$ to $\triangle_2$. Show that $\triangle_1, \triangle_2$ are congruent if and only if they have equal angles. What other conditions for congruence can you find?

(16) With the notation of Question (12), given a polygonal decomposition of S^2 into convex spherical polygons, prove the identity

$$\sum_n (6 - n)F_n = 12 + 2\sum_m (m - 3)V_m.$$

If each face has at least three edges, and at least three edges meet at each vertex, deduce the inequality $3F_3 + 2F_4 + F_5 \geq 12$.

The surface of a football is decomposed into (convex) spherical hexagons and pentagons, with precisely three faces meeting at each vertex. How many pentagons are there? Demonstrate the existence of such a decomposition with each vertex contained in precisely one pentagon.

Note to the reader: You should look at all the questions up to Question 12, and then any further questions you have time for.