Example Sheet 2, Geometry 2007 pmhw@dpmms.cam.ac.uk

(1) Let U be an open subset of $\mathbf{R}^2$ equipped with a Riemannian metric $E\,du^2 + 2F\,du\,dv + G\,dv^2$. For P any point of U, prove that there exists $\lambda > 0$ and an open neighbourhood V of P in U such that

$$(E - \lambda)\,du^2 + 2F\,du\,dv + (G - \lambda)\,dv^2$$

is a Riemannian metric on V.

Defining the distance between two points of U to be the infinum of the lengths of curves joining them, prove that this defines a metric on U. Give an example where this distance is not realized as the length of any curve joining P to Q.

(2) We define a Riemannian metric on the unit disc $D \subset \mathbf{C}$ by $(du^2 + dv^2)/(1 - (u^2 + v^2))$. Prove that the diameters (monotonically parametrized) are length minimizing curves for this metric. Defining the distance between two points of D as in Question 1, show that the distances in this metric are bounded, but that the areas are unbounded.

(3) Suppose that z_1, z_2 are points in the upper half-plane, and suppose the hyperbolic line through z_1 and z_2 meets the real axis at points z_1^* and z_2^*, where z_1 lies on the hyperbolic line segment $z_1^* z_2$, and where one of z_1^* and z_2^* might be ∞. Show that the hyperbolic distance $\rho(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1^*, z_1, z_2, z_2^*, taken in an appropriate order.

(4) With z_1, z_2 points in the upper half-plane,, use the formula for the metric on the unit disc to prove that the hyperbolic distance $\rho(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|$.

(5) Let C denote a hyperbolic circle of hyperbolic radius ρ in the upper half-plane model of the hyperbolic plane; show that C is also a Euclidean circle. If C has hyperbolic centre ic, find the radius and centre of C regarded as a Euclidean circle.

(6) Show that a hyperbolic circle of hyperbolic radius ρ has hyperbolic area

$$A = 2\pi(\cosh(\rho) - 1).$$

[Note that A grows like πe^ρ, while a Euclidean circle of radius ρ has area $\pi\rho^2$.]

(7) Given two points P and Q in the hyperbolic plane, show that the locus of points equidistant from P and Q is a hyperbolic line, the perpendicular bisector of the hyperbolic line segment from P to Q.

(8) Given two hyperbolic lines meeting at a point, show that the locus of points equidistant from the two lines forms two further hyperbolic lines through the point. Show that in a hyperbolic triangle, none of whose vertices are at infinity, the angle bisectors are concurrent.

(9) Show that any isometry g of the disc model D for the hyperbolic plane is **either** of the form (for some $a \in D$ and $0 \le \theta < 2\pi$):

$$g(z) = e^{i\theta}\frac{z - a}{1 - \bar{a}z},$$

or of the form

$$g(z) = e^{i\theta}\frac{\bar{z} - a}{1 - \bar{a}\bar{z}}.$$

(10) Prove that a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$ has area

$$(n-2)\pi - \sum \alpha_i.$$

Show that for every $n \geq 3$ and every α with $0 < \alpha < (1 - \frac{2}{n})\pi$, there is a regular n-gon all of whose angles are α.

(11) Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the perpendicular is unique.

(12) Fix a point P on the boundary of D, the disc model of the hyperbolic plane. Give a description of the curves in D that are orthogonal to every hyperbolic line that passes through P.

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(13) Let l be a hyperbolic line and P a point on l. Show that there is a unique hyperbolic line l' through P making an angle α with l. If α, β are positive numbers with $\alpha + \beta < \pi$, show that there exists a hyperbolic triangle (one vertex at infinity) with angles 0, α and β. For any positive numbers α, β, γ, with $\alpha + \beta + \gamma < \pi$, show that there exists a hyperbolic triangle with these angles. [Hint: For the last part, you may need a continuity argument.]

(14) For arbitrary points z, w in $\mathbf{C}$, prove the identity

$$|1 - \bar{z}w|^2 = |z - w|^2 + (1 - |z|^2)(1 - |w|^2).$$

Given points z, w in the unit disc model of the hyperbolic plane, prove the identity

$$\sinh^2(\frac{1}{2}\rho(z, w)) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)},$$

where ρ denotes the hyperbolic distance.

(15) Let $\triangle$ be a hyperbolic triangle, with angles α, β, γ, and sides of length a, b, c (the side of length a being opposite the vertex with angle α, and similarly for b and c). Using the result from Question 14, and the Euclidean cosine formula, prove the hyperbolic cosine formula, namely

$$\cosh c = \cosh a \ \cosh b \ - \ \sinh a \ \sinh b \cos \gamma.$$

(16) Assuming the hyperbolic cosine formula for hyperbolic triangles, prove the hyperbolic sine formula, namely

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}.$$

[Hint: Reduce this to showing that $(\sinh a \ \sinh b)^2 - (\cosh a \ \cosh b - \cosh c)^2$ is symmetric in a, b and c.]

Deduce that $a \leq b \leq c$ if and only if $\alpha \leq \beta \leq \gamma$.

Note to the reader : You should look at all the questions up to Question 12, and then any further questions you have time for.