Example Sheet 1, Geometry 2007 pmhw@dpmms.cam.ac.uk

(1) Suppose that H is a hyperplane in Euclidean n-space $\mathbf{R}^n$ defined by $\mathbf{u} \cdot \mathbf{x} = c$ for some unit vector $\mathbf{u}$ and constant c. The reflection in H is the map from $\mathbf{R}^n$ to itself given by $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u} - c)\mathbf{u}$. Show that this is an isometry. Letting P, Q be points of $\mathbf{R}^n$, show that there is a reflection in some hyperplane that maps P to Q.

(2) Suppose that l_1 and l_2 are non-parallel lines in the Euclidean plane $\mathbf{R}^2$, and that r_i denotes the reflection of $\mathbf{R}^2$ in the line l_i, for $i = 1, 2$. Show that the composite $r_1 r_2$ is a rotation of $\mathbf{R}^2$, and describe (in terms of the lines l_1 and l_2) the resulting fixed point and the angle of rotation.

(3) Let $R(P, \theta)$ denote the clockwise rotation of $\mathbf{R}^2$ through an angle θ about a point P. If A, B, C are the vertices, labelled clockwise, of a triangle in $\mathbf{R}^2$, prove that $R(A, \theta)R(B, \phi)R(C, \psi)$ is the identity if and only if $\theta = 2\alpha$, $\phi = 2\beta$ and $\psi = 2\gamma$, where α, β, γ denote the angles at, respectively, the vertices A, B, C of the triangle ABC.

(4) Show from first principles that a (continuous) curve of shortest length between two points in Euclidean space is a straight line segment, parametrized monotonically.

(5) Prove that any isometry of the unit sphere is induced from an isometry of $\mathbf{R}^3$ which fixes the origin. Prove that any matrix $A \in O(3, \mathbf{R})$ is the product of at most three reflections in planes through the origin. Deduce that an isometry of the unit sphere can be expressed as the product of at most three reflections in spherical lines. What isometries are obtained from the product of two reflections? What isometries are obtained from the product of three reflections?

(6) By repeatedly applying the result from Question 1, when P is either $\mathbf{0}$ or one of the standard basis vectors of $\mathbf{R}^n$, deduce that any isometry T of $\mathbf{R}^n$ can be written as a composition of at most $n + 1$ reflections.

(7) Suppose that P is a point on the unit sphere S^2. For fixed ρ, with $0 < \rho < \pi$, the *spherical circle* with centre P and radius ρ is the set of points $Q \in S^2$ whose spherical distance from P is ρ. Prove that a spherical circle of radius ρ on S^2 has circumference $2\pi \sin \rho$ and area $2\pi(1 - \cos \rho)$.

(8) Given a spherical line l on the sphere S^2 and a point P not on l, show that there is a spherical line l' passing through P and intersecting l at right-angles. Prove that the minimum distance $d(P, Q)$ of P from a point Q on l is attained at one of the two points of intersection of l with l', and that l' is unique if this minimum distance is less than $\pi/2$.

(9) Let $\pi : S^2 \to \mathbf{C}_\infty$ denote the stereographic projection map. Show that the spherical circles on S^2 biject under π with the circles and straight lines on $\mathbf{C}$.

(10) Show that any Möbius transformation $T \neq 1$ on $\mathbf{C}_\infty$ has one or two fixed points. Show that the Möbius transformation corresponding (under the stereographic projection map) to a rotation of S^2 through a non-zero angle has exactly two fixed points z_1 and z_2, where $z_2 = -1/\bar{z}_1$. If now T is a Möbius transformation with two fixed points z_1 and z_2 satisfying $z_2 = -1/\bar{z}_1$, prove that **either** T corresponds to a rotation of S^2, **or** one of the fixed points, say z_1, is an *attractive* fixed point, i.e. for $z \neq z_2$, $T^n z \to z_1$ as $n \to \infty$.

(11) Prove that Möbius transformations of $\mathbf{C}_\infty$ preserve cross-ratios. If $u, v \in \mathbf{C}$ correspond to points P, Q on S^2, and d denotes the angular distance from P to Q on S^2, show that $-\tan^2 \frac{1}{2}d$ is the cross ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in an appropriate order (which you should specify).

(12) Let M denote a convex polyhedron in $\mathbf{R}^3$, with F faces, E edges and V vertices. Using the Gauss–Bonnet theorem for spherical triangles, prove that $F - E + V = 2$. Now let M denote one of the five regular solids, and consider the radial projection map from the centre of M to the sphere S^2. In each case, identify the spherical polygon on S^2 (specifying the angles) which occurs as the image of a face of M.

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(13) For every spherical triangle $\triangle = ABC$, show that $a < b + c$, $b < c + a$, $c < a + b$ and $a + b + c < 2\pi$. Conversely, show that for any three positive numbers a, b, c less than π satisfying the above conditions, we have $\cos(b + c) < \cos a < \cos(b - c)$, and that there is a spherical triangle (unique up to isometries of S^2) with those sides.

(14) A spherical triangle $\triangle = ABC$ has vertices given by unit vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathbf{R}^3$, sides of length a, b, c, and angles α, β, γ (where the side opposite vertex A is of length a and the angle at A is α, etc.). The *polar* triangle $A'B'C'$ is defined by the unit vectors in the directions $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{B}$. Prove that the sides and angles of the polar triangle are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$, and $\pi - a$, $\pi - b$, $\pi - c$ respectively. Deduce the formula

$$\sin\alpha \, \sin\beta \, \cos c = \cos\gamma + \cos\alpha \, \cos\beta \, .$$

(15) Two spherical triangles $\triangle_1, \triangle_2$ on a sphere S^2 are said to be *congruent* if there is an isometry of S that takes $\triangle_1$ to $\triangle_2$. Show that $\triangle_1, \triangle_2$ are congruent if and only if they have equal angles. What other conditions for congruence can you find?

(16) For a cube centred on the origin in $\mathbf{R}^3$, show that the rotation group is isomorphic to S_4. What is the full symmetry group? How many of the isometries is this group are rotated reflections (and not pure reflections)? Describe these rotated reflections geometrically, by identifying the axes of rotation and the angles of rotation.

Note to the reader : You should look at all the questions up to Question 12, and then any further questions you have time for.