Example Sheet 3, Geometry 2006 pmhw@dpmms.cam.ac.uk

(1) Show the tangent space to S^2 at a point $P = (x, y, z) \in S^2$ is the plane normal to the vector $\overrightarrow{OP}$, where O denotes the origin.

(2) Let V be the open subset $\{0 < u < \pi,\ 0 < v < 2\pi\}$, and $\sigma : V \to S^2$ be given by

$$\sigma(u, v) = (\sin u\ \cos v, \sin u\ \sin v, \cos u).$$

Prove that σ defines a smooth parametrization on a certain open subset of S^2. [*You may assume that* $\cos^{-1}$ *is continuous on* $(-1, 1)$, *and that* $\tan^{-1}$, $\cot^{-1}$ *are continuous on* $(-\infty, \infty)$.]

(3) Show the stereographic projection map $\pi : S \setminus \{N\} \to \mathbf{C}$, where N denotes the north pole, defines a chart. Check that the spherical metric on $S \setminus \{N\}$ corresponds under π to the Riemannian metric on $\mathbf{C}$ given by

$$4(dx^2 + dy^2)/(1 + x^2 + y^2)^2.$$

(4) For an embedded circular cylinder S in $\mathbf{R}^3$, show that the first fundamental form corresponds to a locally Euclidean Riemannian metric on S. Identify the geodesics on S.

(5) Given a smooth curve $\Gamma : [0, 1] \to S$ on an abstract surface S with a Riemannian metric, show that the length l is unchanged under reparametrizations of the form $f : [0, 1] \to [0, 1]$, with $f'(t) > 0$ for all $t \in [0, 1]$. Prove that there exists such a reparametrization $\tilde{\Gamma} = \Gamma \circ f$ for which $\|d\tilde{\Gamma}/dt\|$ is constant, namely l.

(6) Let T denote the embedded torus in $\mathbf{R}^3$ obtained by revolving around the z-axis the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane. Using the formal definition of area in terms of a parametrization, calculate the surface area of T.

(7) If one places S^2 inside a (vertical) circular cylinder of radius one, prove that the radial (horizontal) projection map from S^2 to the cylinder preserves areas (this is usually known as *Archimedes Theorem*). Deduce the existence of an atlas on S^2, for which the charts all preserve areas and the transition functions have derivatives with determinant one.

(8) Show that a 2-holed torus may be obtained topologically by suitably identifying the sides of a regular octagon. Indicate briefly how to extend your argument to show that a g-holed torus may be obtained topologically by suitably identifying the sides of a regular $4g$-gon?

(9) Let $S \subset \mathbf{R}^3$ be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If $V \subset \mathbf{R}^2$ denotes the region $u^2/a^2 + v^2/b^2 < 1$, show that the map

$$\sigma(u, v) = \left(u,\ v,\ c\,(1 - u^2/a^2 - v^2/b^2)^{\frac{1}{2}}\right)$$

determines a smooth parametrization of a certain open subset of S. Prove that the Gaussian curvatures at the points $(a, 0, 0), (0, b, 0), (0, 0, c)$ are all equal if and only if S is a sphere.

(10) For a surface of revolution S, corresponding to a curve $\eta : (a,b) \to \mathbf{R}^3$ given by $\eta(u) = (f(u), 0, g(u))$, where η is parametrized in such a way that $\|\eta'\| = 1$, prove that the second fundamental form at a given point is given by

$$(f'g'' - f''g')du^2 \ + \ fg'dv^2.$$

Deduce that the Gaussian curvature K is given by the formula $K = -f''/f$.

(11) Using the results from the previous question, calculate the Gaussian curvature K of the unit sphere. For the embedded torus, as defined in Question 6, identify those points at which $K = 0$, $K > 0$ and $K < 0$. Verify the global Gauss–Bonnet theorem on the embedded torus.

(12) Let S be an embedded surface in $\mathbf{R}^3$ which is closed and bounded. By considering the smallest closed ball centred on the origin which contains S, or otherwise, show that the Gaussian curvature must be strictly positive at some point of S. Deduce that the locally Euclidean metric on the torus T cannot be realised as the first fundamental form for some smooth embedding of T in $\mathbf{R}^3$.

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(13) Show that Mercator's parametrization of the sphere (minus poles)

$$\sigma(u,v) = (\operatorname{sech} u \ \cos v, \operatorname{sech} u \ \sin v, \tanh u)$$

determines a chart (on the complement of a longitude) which preserves angles and sends meridians and parallels on the sphere to straight lines in the plane.

(14) Show that a g-holed torus ($g > 1$) may be given the structure of an abstract surface with a Riemannian metric, in such a way that it is locally isometric to the hyperbolic plane. [For this question, you will need an argument similar to Q8 on this sheet, and you will need the result from Q10 on Example Sheet 2.]

(15) Let $f(u) = e^u$, $g(u) = (1 - e^{2u})^{\frac{1}{2}} - \cosh^{-1}(e^{-u})$, where $u < 0$, and S be the surface of revolution corresponding to the curve $\eta : (-\infty, 0) \to \mathbf{R}^3$ given by $\eta(u) = (f(u), 0, g(u))$. Show that S has constant Gauss curvature -1; S is called the *pseudosphere*. By considering coordinates v and $w = e^{-u}$ on S, show that the pseudosphere is locally isometric to the open subset of the upper half-plane model of the hyperbolic plane given by $\operatorname{Im}(z) > 1$. Identify the geodesics in this model corresponding to the meridians in S. Show that any other geodesic in S has two distinct points of the circle $x^2 + y^2 = 1, z = 0$ as limit points.

[It is a theorem that there are no complete embedded surfaces in $\mathbf{R}^3$ with constant negative Gauss curvature, and so in particular we cannot realise all of the hyperbolic plane as an embedded surface.]

Note to the reader : You should look at all the questions up to Question 12, and then any further questions you have time for.