Mock Tripos Questions, Geometry 2005 pmhw@dpmms.cam.ac.uk

The following are Tripos style questions on the Geometry course, Section I consisting of short questions and Section II of long questions. Questions 1,2,4 and 6 are are based on recent Tripos questions.

SECTION I

(1) Explain how to obtain, from any convex polyhedron, a decomposition of the surface of the unit sphere into spherical polygons. State the Gauss–Bonnet theorem for the area of a spherical triangle.

Show that any regular polyhedron whose faces are pentagons has the same number of faces, edges and vertices as the dodecahedron.

(2) Using the Riemannian metric $(dx^2 + dy^2)/y^2$, define the length of a curve and the area of a region in the upper half-plane $H = \{x + iy \,:\, y > 0\}$.

Find the hyperbolic area of the region $\{(x, y) \in H \,:\, 0 < x < 1,\ y > 1\}$.

(3) Let V be an open subset of $\mathbf{R}^2$, with a Riemannian metric $E\,du^2 + 2F\,du\,dv + G\,dv^2$. State (without proof) the equations for a smooth curve $\gamma : [0, 1] \to V$ to be geodesic.

Consider now the smooth parametrization $\sigma : (0, \pi) \times (0, 2\pi) \to U \subset S^2$ given by

$$\sigma(\rho, \theta) = (\sin\rho\cos\theta, \sin\rho\sin\theta, 1 - \cos\rho),$$

where U is the open subset of the unit sphere given as the complement of the longitude $y = 0,\ x \geq 0$. Find the first fundamental form in these coordinates. Verify that the curves $\Gamma(t)$ with both θ and $d\rho/dt$ constant are geodesics, and that these correspond to the spherical lines through $(0, 0, 1)$.

SECTION II

(4) Show that every isometry of Euclidean space $\mathbf{R}^3$ is a composition of at most four reflections in planes, and that an isometry which fixes the origin is a composition of at most three reflections.

Describe geometrically the isometries g of $\mathbf{R}^3$ fixing the origin which require three reflections. If g does not correspond to $-I$, show that there is a unique line through the origin sent to itself under g.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedon. For those which are not pure reflections, identify a line through the centre of the tetrahedron which is mapped to itself.

(5) Describe geometrically the stereographic projection map $\phi : S^2 \to \mathbf{C}_\infty$, and find a formula for it. Let A denote the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

representing a rotation of S^2 through an angle $\pi/2$ about the y-axis. Show that A corresponds under ϕ to a Möbius tranformation g, where $g(\zeta) = (\zeta - 1)/(\zeta + 1)$. Prove that any rotation of S^2 corresponds under ϕ to a Möbius tranformation of $\mathbf{C}_\infty$. Deduce that the image under ϕ of a spherical circle on S^2 is a circle or straight line in $\mathbf{C}_\infty$.

(6) Show that for every hyperbolic line L in the hyperbolic plane H there is a *unique* isometry of H which is the identity on L but not on all of H. Call this the *reflection* R_L.

Prove that every isometry of H is a composition of reflections.

(7) Let $S \subset \mathbf{R}^3$ be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. If $V \subset \mathbf{R}^2$ denotes the region $u^2/a^2 + v^2/b^2 < 1$, show in detail that the map

$$\sigma(u, v) = \big(u,\ v,\ c\,(1 - u^2/a^2 - v^2/b^2)^{\frac{1}{2}}\big)$$

determines a smooth parametrization of a certain open subset of S.

Calculate from first principles the first and second fundamental forms (in these coordinates) at the point $P = \frac{1}{\sqrt{3}}(a, b, c) \in S$. Prove that the Gaussian curvature at the point P is given by the formula

$$K = \frac{9a^2b^2c^2}{(a^2b^2 + b^2c^2 + c^2a^2)^2}.$$

Prove furthermore that the Gaussian curvatures at the points $(a, 0, 0), (0, b, 0), (0, 0, c)$ are all equal if and only if S is a sphere.