Complex Analysis IB: 2025-26 – Sheet 2

Let $B(a;\epsilon)$ denote the open ball $\{z \in \mathbb{C} : |z-a| < \epsilon\}$.

1. The Weierstrass approximation theorem states that any continuous function $f : I \to \mathbb{R}$ on a closed bounded connected subset $I \subset \mathbb{R}$ can be uniformly approximated by polynomials. Can any continuous function $\phi : J \to \mathbb{C}$ on a closed bounded connected subset $J \subset \mathbb{C}$ be uniformly approximated by polynomials? Justify your answer.

2. (i) For $\alpha \in \mathbb{C}$, use the Cauchy integral formula to compute
$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz.$$
(ii) By considering suitable complex integrals, show that
$$\int_0^{\pi} \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1-r^2}\ \forall\, r \in (0,1); \quad \text{and} \quad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta \;=\; 2\pi.$$

3. Let f be an entire function.
(i) If $f(z)/z \to 0$ as $|z| \to \infty$, prove that f is constant. (This strengthens Liouville's theorem.)
(ii) If for some $a \in \mathbb{C}$ and $\epsilon > 0$, f never takes values in $B(a;\epsilon)$, show that f is constant.
(iii) If $f = u + iv$ and $|u| > |v|$ throughout $\mathbb{C}$, show that f is constant.

4. Let U be a domain and $f : U \to \mathbb{C}$ be holomorphic. If the real part $\mathrm{Re}(f)$ has an interior local maximum at $a \in U$, show that f is constant.

5. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1+|z|)^k$ for all z.
(ii) Show that an entire function f is a non-constant polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.
(iii) Let f be a function which is holomorphic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

6. (i) (Schwarz's Lemma) Let f be holomorphic on the open unit disk $B(0;1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.
(ii) Use Schwarz's Lemma to prove that any conformal equivalence from the unit disk to itself is given by a Möbius transformation.

7. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:
$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \bigm| |z| > 2\}.$$

8. Classify the singularities of each of the following functions:

$$\frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

9. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(1/n) = 1/n$ for each $n \in \mathbb{Z}_{>0}$, show that $f(z) = z$.

   (ii) Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(n) = n^2$ for every $n \in \mathbb{Z}$, must $f(z) = z^2$?

   (iii) Let f be holomorphic on $B(0;2)$. Show that $f(1/n) \neq 1/(n+1)$ for some $n \in \mathbb{Z}_{>0}$.

10. (i) Give an example of an infinitely differentiable function $f : (-1,1) \to \mathbb{R}$ which can be extended to a holomorphic function on a domain $(-1,1) \subset U \subset \mathbb{C}$, but for which one cannot take U to be the open unit disc $B(0;1)$.

    (ii) Give an example of an infinitely differentiable function $f : (-1,1) \to \mathbb{R}$ which is not the restriction of any holomorphic function defined on a domain $(-1,1) \subset U \subset \mathbb{C}$.

    (iii) Prove that the integral $\int_0^\infty e^{-zt}\sin(t)\,dt$ converges for $\mathrm{Re}(z) > 0$ and defines a holomorphic function in that half-plane. Prove furthermore that the resulting holomorphic function admits an analytic continuation to $\mathbb{C}\backslash\{\pm i\}$.

    (iv) Prove that the series $\sum_{n=0}^{\infty} z^{(2^n)}$ defines a holomorphic function on the disc $B(0;1)$ which admits no analytic continuation to any larger domain $B(0;1) \subsetneq U \subset \mathbb{C}$.

11. (Casorati-Weierstrass theorem) Let f be holomorphic on $B(a;r) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in B(a;r)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

    [A much harder theorem of Picard says that in any neighbourhood of an essential singularity, a holomorphic function takes *every* complex value except possibly one.]

12. Let $f\colon B(a;R) \setminus \{a\} \to \mathbb{C}$ be holomorphic. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\mathrm{Re} f(z) < M$ for $z \in B(a;R)$, then f has a removable singularity at $z = a$.

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