Complex Analysis IB: 2025-26 – Sheet 1

1. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that T can be written $Tz = Az + B\bar{z}$ for unique $A, B \in \mathbb{C}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : U \to \mathbb{C}$ be a holomorphic function defined on a domain U. Show that f is constant if any of its real part, modulus or argument is constant.

   (ii) Find all entire functions of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

   (iii) Find all entire functions which have real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

   $$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

   Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4. (i) Verify directly that e^z and $\cos z$ satisfy the Cauchy-Riemann equations everywhere.

   (ii) Find the set of $z \in \mathbb{C}$ for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

   (iii) Find the zeros of $1 + e^z$ and of $\cosh z$.

5. (i) If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce

   $$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad \forall\, z \in \mathbb{C}.$$

   (ii) Defining $z^\alpha = e^{\alpha \operatorname{Log} z}$, for Log the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz} z^\alpha = \alpha\, z^{\alpha - 1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold ?

6. Find conformal equivalences between the following pairs of domains:

   (i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc D;

   (ii) the lune $\{z \in \mathbb{C} : |z - 1| < \sqrt{2} \text{ and } |z + 1| < \sqrt{2}\}$ and the open unit disk D;

   (iii) the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.

   By considering a suitable bounded solution of the Laplace equation $u_{xx} + u_{yy} = 0$ on the strip S, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant (it need not be continuous at the origin).

7. (i) Show that the general Möbius transformation which takes the unit disc to itself has the form $z \mapsto \lambda \frac{z - a}{\bar{a}z - 1}$, with $|a| < 1$, $|\lambda| = 1$. [*Hint: show that these maps* do *send the unit disc to itself, and note that the composition of two Möbius maps preserving the unit disc is another such.*]

   (ii) Find a Möbius transformation taking the region between $\{|z| = 1\}$ and $\{|z - 1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: One can use (i).*]

8. Prove that the following series converge uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n}e^{-nz}, \quad \text{on } \{0 < \mathrm{Re}(z)\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \quad \text{on } \left\{|z| < \frac{1}{2}\right\}.$$

9. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

10. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivative) on the domains indicated:

$$(a)\ \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1); \qquad (b)\ \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$

11. Let $U \subset \mathbb{C}$ be a domain and $u : U \to \mathbb{R}$ a C^2-harmonic function. Show that if $z_0 \in U$ and $B = B(z_0, r) \subset U$ is a disc, there is a holomorphic function $f : B \to \mathbb{C}$ with $u = \mathrm{Re}(f)$ on B. Give an example to show that this need not be true globally (i.e. there is a domain U and a harmonic function on U which is not the real part of any holomorphic function on U).

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