Complex Analysis Example Sheet 3 **Lent 2025**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be emailed to me at `hkrieger@dpmms.cam.ac.uk`.

1. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$, and $|w-a|<r<R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}}\,dz.$$

2. Let $g(z)=p(z)/q(z)$ be a rational function with $\deg(q)\geq \deg(p)+2$. Show that the sum of the residues of f at all its poles equals zero.

3. Evaluate the following integrals:

$$(a)\quad \int_0^\pi \frac{d\theta}{4+\sin^2\theta} \qquad\qquad (b)\quad \int_0^\infty \sin x^2\,dx$$

$$(c)\quad \int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx \qquad\qquad (d)\quad \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$$

4. For $\alpha\in(-1,1)$ with $\alpha\neq 0$, compute

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1}\,dx$$

5. Prove the following refinement of the Fundamental Theorem of Algebra: let $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ be a polynomial of degree n, and write $A=\max\{|a_i|\}$. Then p has n roots (counted with multiplicity) in the disk $|z|<A+1$.

6. Let $p(z)=z^5+z$. Find all z such that $|z|=1$ and $\operatorname{Im} p(z)=0$. Calculate $\operatorname{Re} p(z)$ for such z. Hence sketch the curve $p\circ\gamma$, where $\gamma(t)=e^{2\pi i t}$ and use your sketch to determine the number of z (counted with multiplicity) such that $|z|<1$ and $p(z)=x$ for each real number x.

7. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1\pm i)(N+1/2)$. Show that there exists $C>0$ such that for every N, $|\cot\pi z|<C$ on γ_N.

(ii) By integrating $\dfrac{\pi\cot\pi z}{z^2+1}$ around γ_N, show that

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1} = \frac{1+\pi\coth\pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty}(-1)^n/(n^2+1)$.

8. (i) Show that $z^4 + 12z + 1$ has exactly three zeroes with $1 < |z| < 4$.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{ z \mid \text{Re(z)} < 0 \}$.

9. Show that the equation $z \sin z = 1$ has only real solutions.
Hint: Find the number of real roots in the interval $[-(n + 1/2)]\pi, (n + 1/2)\pi]$ *and compare with the number of zeros of* $z \sin z - 1$ *in the square box* $\{|\text{Re}(z)|, |\text{Im}(z)| < (n + 1/2)\pi\}$.

10. (i) Let $w \in \mathbb{C}$, and let $\gamma, \delta\colon [0, 1] \to \mathbb{C}$ be closed curves such that for all $t \in [0, 1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $I(\gamma; w) = I(\delta; w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.

11. Let U be a domain, $f : U \to \mathbb{C}$ holomorphic, and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) := \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function on a neighborhood of $f(a)$ which is inverse to f.