Complex Analysis Example Sheet 2 **Lent 2023**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be emailed to me at `hkrieger@dpmms.cam.ac.uk`.

1. (i) Use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz$$

where $\alpha \in \mathbb{C}$.

(ii) By considering the real part of a suitable complex integral, show that if $r \in (0, 1)$,

$$\int_0^{\pi} \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2} \qquad \text{and} \qquad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta = 2\pi.$$

2. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \mid |z| > 2\}.$$

3. Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if any one of the following conditions hold, then f is constant:

(i) $f(z)/z \to 0$ as $|z| \to \infty$.

(ii) There exists $b \in \mathbb{C}$ and $\epsilon > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - b| > \epsilon$.

(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

5. Let $f : D(a, r) \to \mathbb{C}$ be holomorphic, and suppose that $z = a$ is a local maximum for $\mathrm{Re}(f)$. Show that f is constant.

6. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.

(ii) Show that an entire function f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

(iii) Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

7. (i) (Schwarz's Lemma) Let f be analytic on $D(0, 1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.

(ii) Use Schwarz's Lemma to prove that any conformal equivalence from $D(0, 1)$ to itself is given by a Möbius transformation.

8. (i) Let f be an entire function such that for every positive integer n, $f(1/n) = 1/n$. Show that $f(z) = z$.

(ii) Let f be an entire function with $f(n) = n^2$ for every $n \in \mathbb{Z}$. Must $f(z) = z^2$?

(iii) Let f be holomorphic on $D(0, 2)$. Show that for some integer $n > 0$, $f(1/n) \neq 1/(n + 1)$.

9. (Casorati-Weierstrass theorem) Let f be holomorphic on $D(a, R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a, R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

10. Let $D \subset \mathbb{C}$ be a simply-connected domain which does not contain 0. Show that there exists a branch of the logarithm on D.

11. Show that the power series $\sum_{n=1}^{\infty} z^{2^n}$ defines an analytic function f on $D(0, 1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0, 1)$.