Complex Analysis Example Sheet 3 **Lent 2023**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be emailed to me at `hkrieger@dpmms.cam.ac.uk`.

1. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$, and $|w-a|<r<R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}}\,dz.$$

2. Let $g(z) = p(z)/q(z)$ be a rational function with $\deg(q) \geq \deg(p)+2$. Show that the sum of the residues of f at all its poles equals zero.

3. Evaluate the following integrals:

$$(a)\quad \int_0^\pi \frac{d\theta}{4+\sin^2\theta} \qquad\qquad (b)\quad \int_0^\infty \sin x^2\,dx$$

$$(c)\quad \int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx \qquad\qquad (d)\quad \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$$

4. For $\alpha \in (-1,1)$ with $\alpha \neq 0$, compute

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1}\,dx$$

5. Prove the following refinement of the Fundamental Theorem of Algebra: let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, and write $A = \max\{|a_i|\}$. Then p has n roots (counted with multiplicity) in the disk $|z| < A+1$.

6. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z. Hence sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$ and use your sketch to determine the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each real number x.

7. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N+1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$ around γ_N, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1+\pi\coth\pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty}(-1)^n/(n^2+1)$.

8. (i) Show that $z^4 + 12z + 1$ has exactly three zeroes with $1 < |z| < 4$.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \operatorname{Re}(z) < 0 \,\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $|z| = 3/2$.

9. Show that the equation $z \sin z = 1$ has only real solutions.

Hint: Find the number of real roots in the interval $[-(n+1/2)]\pi, (n+1/2)\pi]$ *and compare with the number of zeros of* $z \sin z - 1$ *in the square box* $\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)| < (n+1/2)\pi\}$.

10. (i) Let $w \in \mathbb{C}$, and let $\gamma, \delta \colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $I(\gamma; w) = I(\delta; w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.

(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

11. Let U be a domain, $f : U \to \mathbb{C}$ holomorphic, and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) := \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w} \, dz$$

defines a holomorphic function on a neighborhood of $f(a)$ which is inverse to f.

The following integrals are not part of the example sheet, but may be useful for revision or fun.

1. For $a, m \in \mathbb{R}^+$, evaluate
$$\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2 + x^2)} \, dx.$$

2. For $a \in (0,1)$, evaluate
$$\int_0^{2\pi} \frac{\cos^3 3t}{1 - 2a \cos t + a^2} \, dt.$$

3. Using a 'dog-bone contour', evaluate
$$\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{1 + x^2} \, dx.$$

4. For $t \in \mathbb{R}$, evaluate
$$\int_{-\infty}^{\infty} e^{-ax^2} e^{-itx} \, dx \qquad \text{where } a > 0,\ t \in \mathbb{R}.$$

5. For $t \in \mathbb{R}$, evaluate
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx} \, dx.$$

6. Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that
$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2} = \frac{\pi}{2}(\beta - \alpha),$$
and deduce the value of
$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 \, dx.$$