COMPLEX ANALYSIS EXAMPLES 3, LENT 2020

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1. Let U be an open subset of $\mathbb{C}$ and let $\gamma_0, \gamma_1 : [0,1] \to U$ be (piecewise C^1) closed curves. Recall that γ_0 is said to be homotopic to γ_1 in U if there is a continuous map $H : [0,1] \times [0,1] \to U$ with $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$ for $0 \leq t \leq 1$ and $H(s,0) = H(s,1)$ for $0 \leq s \leq 1$. Prove that if γ_0 is homotopic to γ_1, then $I(\gamma_0; w) = I(\gamma_1; w)$ for each $w \in \mathbb{C} \setminus U$. [Hint: For each s, $\gamma_s(t) = H(s,t)$, $0 \leq t \leq 1$ is a closed continuous curve; consider, for a sufficiently large fixed positive integer n, and $k \in \{0, 1, 2, \ldots, n\}$, the polygonal curves $\widetilde{\gamma}_k$ defined by $\widetilde{\gamma}_k(t) = \gamma_{\frac{k}{n}}(\frac{j}{n})(nt + 1 - j) + \gamma_{\frac{k}{n}}(\frac{j-1}{n})(j - nt)$ for $j \in \{1, 2, \ldots, n\}$ and $\frac{j-1}{n} \leq t \leq \frac{j}{n}$. Use the result of Q11(i), ex. sheet 2.]

Deduce that if a piecewise C^1 curve in U is null-homotopic (i.e. homotopic in U to a constant curve), then it is homologous to zero in U. Draw a picture of a domain U and a curve in U that is homologous to zero in U but is not null-homotopic in U.

2. Recall that we defined a domain $U \subset \mathbb{C}$ to be a simply connected if every closed piecewise C^1 curve in U is homologous to zero in U, and proved Cauchy's theorem for such domains. Use Cauchy's theorem to show that if U is simply connected and if f is a nowhere vanishing holomorphic function on U, then f admits a holomorphic square-root (i.e. there is a holomorphic function h such that $h^2(z) = f(z)$ for every $z \in U$.)

The key ingredient of a standard proof of the Riemann mapping theorem is to show that if a domain $U \neq \mathbb{C}$ has the property that every nowhere zero holomorphic $f : U \to \mathbb{C}$ admits a holomorphic square-root, then U is homeomorphic (in fact conformally equivalent) to the open unit disk. Assuming this, deduce that every closed piecewise C^1 curve in a simply connected domain U is null-homotopic in U (in other words, U is simply connected also in the sense of algebraic topology; so with the result of Q1 above, the two notions of simple connectivity are equivalent).

3. The Weierstrass approximation theorem in real analysis says that every continuous function $f : I \to \mathbb{R}$ on a compact interval $I \subset \mathbb{R}$ is the uniform limit of a sequence of polynomials. The direct analogue of this to the complex setting (obtained by replacing $\mathbb{R}$ with $\mathbb{C}$ and I with a compact set $K \subset \mathbb{C}$) is false, even if we make a suitable holomorphicity assumption on f. Construct, for any given compact set $K \subset \mathbb{C}$ with $\mathbb{C} \setminus K$ not connected, a function f that is holomorphic on an open set containing K such that f is not the uniform limit on K of a sequence of complex polynomials. [Hint: you may wish to generalise the idea of Q 12(ii) in sheet 1 for the construction, and use the global maximum principle to prove it works.] Look up, on the other hand, Runge's theorem and Mergelyan's theorem!

4. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a, R)$, and $|w - a| < r < R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D(a,r)} \frac{f(z)}{(z - w)^{n+1}} \, dz.$$

5. Prove the following facts often found useful for computing integrals:
(i) *Jordan's lemma:* If f is holomorphic on $\{|z| > r\}$ for some $r > 0$, and if $zf(z)$ is bounded for $|z|$ large, then for each $\alpha > 0$, $\int_{\gamma_R} f(z)e^{i\alpha z}\,dz \to 0$ as $R \to \infty$, where $\gamma_R(t) = Re^{it}$, $0 \le t \le \pi$. [Use the fact that $\frac{\sin t}{t} \ge \frac{2}{\pi}$ for $0 < t \le \frac{\pi}{2}$.]
(ii) If f has a simple pole at a, and if γ_ϵ is the curve $\gamma_\epsilon(t) = a + \epsilon e^{it}$, $\alpha \le t \le \beta$, then $\int_{\gamma_\epsilon} f(z)\,dz \to (\beta - \alpha)i\mathrm{Res}_f(a)$ as $\epsilon \to 0^+$.

6. Evaluate the following integrals:

$$(a) \quad \int_0^\pi \frac{d\theta}{4 + \sin^2\theta}; \qquad (b) \quad \int_0^\infty \sin x^2\,dx;$$

$$(c) \quad \int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx; \qquad (d) \quad \int_0^\infty \frac{\log(x^2+1)}{x^2+1}\,dx.$$

7. For $\alpha \in (-1, 1)$ with $\alpha \ne 0$, compute $\int_0^\infty \frac{x^\alpha}{x^2+x+1}\,dx$.

8. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.
(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$ around γ_N, show that $\sum_{n=0}^\infty \frac{1}{n^2+1} = \frac{1+\pi\coth\pi}{2}$.
(iii) Evaluate $\sum_{n=0}^\infty (-1)^n/(n^2+1)$.

9. Let f be holomorphic in an open set U except at a point $a \in U$ and at a sequence of points $a_n \in U$ converging to a. Suppose that each a_n is a pole of f. Note that a is then a non-isolated singularity. (i) Give an explicit example of such a function f, points a_n and a. (ii) What can you say (in general) about the image $f(U \setminus \{a, a_1, a_2, \ldots\})$?

10. Let f_n be a sequence of holomorphic functions on a domain U converging locally uniformly to a function $f : U \to \mathbb{C}$. If $f_n(z) \ne 0$ for each n and each $z \in U$, show that either $f(z) = 0$ for all $z \in U$ or $f(z) \ne 0$ for all $z \in U$. What if we allow each f_n to have at most k zeros in U for some fixed positive integer k independent of n?

11. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i| : 0 \le i \le n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $|z| < A + 1$.

12. If $f : U \to \mathbb{C}$ is holomorphic and one-to-one, show that $f'(z) \ne 0$ for all $z \in U$.

13. (i) Show that $z^4 + 12z + 1 = 0$ has exactly three zeros with $1 < |z| < 4$.
(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \mathrm{Re}(z) < 0 \,\}$.
(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z : \ |z| = 3/2 \,\}$.

14. Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \le |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

15. Show that the equation $z \sin z = 1$ has only real solutions. [Hint: Find the number of real roots in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$ and compare with the number of zeros of $z \sin z - 1$ is a square box $\{|\mathrm{Re}\, z|, |\mathrm{Im}\, z| < (n+1/2)\pi\}$.]

16. Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

The following integrals are *not* part of the question sheet, but are provided as a starting point for revision, or for the enthusiast.

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2 + x^2)}\, dx$ where $a,\ m \in \mathbb{R}^+$;

(2) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1 - 2a\cos t + a^2}\, dt$ where $a \in (0,1)$;

(3) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1 - x^2}}{1 + x^2}\, dx$ ("dog-bone" contour);

(4) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx$ where $t \in \mathbb{R}$.

(5) By integrating $z/(a - e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log(1 + a)$$

for every $a \in (0,1)$.

(6) Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that

$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx.$$