

COMPLEX ANALYSIS EXAMPLES 2, LENT 2020

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1. Use the Cauchy integral formula to compute $\int_{\partial D(0,2)} \frac{dz}{z^2+1}$ and $\int_{\partial D(0,2)} \frac{dz}{z^2-1}$. Are the answers an accident? Formulate and prove a result for a polynomial with n distinct roots.

2. (i) Use the Cauchy integral formula to compute

$$\int_{\partial D(0,1)} \frac{e^{\alpha z}}{2z^2 - 5z + 2} dz$$

where $\alpha \in \mathbb{C}$.

(ii) By considering suitable complex integrals, show that if $r \in (0, 1)$,

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r \cos \theta + r^2} d\theta = \frac{\pi r^n}{1 - r^2} \quad \text{and} \quad \int_0^{2\pi} \cos(\cos \theta) \cosh(\sin \theta) d\theta = 2\pi.$$

3. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Prove that if any one of the following conditions hold, then f is constant:

(i) $f(z)/z \rightarrow 0$ as $|z| \rightarrow \infty$.

(ii) There exists $b \in \mathbb{C}$ and $\varepsilon > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - b| > \varepsilon$.

(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

4. Let $f: D(a, r) \rightarrow \mathbb{C}$ be holomorphic, and suppose that $z = a$ is a local maximum for $\operatorname{Re}(f)$. Show that f is constant.

5. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z .

(ii) Show that an entire function f is a polynomial of positive degree if and only if $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.

6. Recall that the (global) maximum principle says that if U is a bounded open subset of $\mathbb{C}$ and if f is continuous on $\overline{U}$ (the closure of U in $\mathbb{C}$) and holomorphic in U , then $\sup_{\overline{U}} |f| = \sup_{\partial U} |f|$. Does this hold if U is not assumed to be bounded?

7. (i) (Schwarz's Lemma) Let f be holomorphic on $D(0, 1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c .

(ii) Use Schwarz's Lemma to prove that any conformal equivalence from $D(0, 1)$ to itself is given by a Möbius transformation.

8. Let U be a bounded open subset of $\mathbb{C}$ and let (f_n) be a sequence of continuous functions on $\overline{U}$ such that f_n is holomorphic on U for each n . If (f_n) converges uniformly on $\partial U = \overline{U} \setminus U$, show that (f_n) converges uniformly on $\overline{U}$.

9. (i) Let f be an entire function such that for every positive integer n , $f(1/n) = 1/n$. Show that $f(z) = z$.

- (ii) Let f be an entire function with $f(n) = n^2$ for every $n \in \mathbb{Z}$. Must $f(z) = z^2$?
 (iii) Let f be holomorphic on $D(0, 2)$. Show that for some integer $n > 0$, $f(1/n) \neq 1/(n+1)$.

10. Show that the power series $\sum_{n=1}^{\infty} z^{n!}$ defines an analytic function f on $D(0, 1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0, 1)$. [Hint: consider $z = \exp(2\pi ip/q)$ with p/q rational.]

- 11.** (i) Let $w \in \mathbb{C}$, and let $\gamma, \delta: [0, 1] \rightarrow \mathbb{C}$ be closed curves such that for all $t \in [0, 1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve

$$\sigma(t) = \frac{\delta(t) - w}{\gamma(t) - w}$$

about the origin, show that $I(\gamma; w) = I(\delta; w)$.

- (ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.

(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ , then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U .

- 12.** Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z : |z| < 1\}; \quad \{z : 1 < |z| < 2\}; \quad \{z : |z| > 2\}.$$

- 13.** Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \quad \sin \frac{\pi}{z^2}, \quad \frac{1}{z^2} + \frac{1}{z^2 + 1}, \quad \frac{1}{z^2} \cos \left(\frac{\pi z}{z + 1} \right).$$

- 14.** (Casorati-Weierstrass theorem) Let f be holomorphic on $D(a, R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a, R)$ with $z_n \neq a$ such that $z_n \rightarrow a$ and $f(z_n) \rightarrow b$ as $n \rightarrow \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

- 15.** Let f be a holomorphic function on $D(a, R) \setminus \{a\}$. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\operatorname{Re} f(z) < M$ for $z \in D(a, R) \setminus \{a\}$, then f has a removable singularity at $z = a$.