Part IB COMPLEX ANALYSIS (Lent 2019): Example Sheet 3

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1. Let f be a holomorphic function on a punctured disc $D^*(a,R) = \{0 < |z-a| < R\}$ and let γ be a closed curve in $D^*(a,R)$. Show that

$$\int_\gamma f(z)\,dz = 2\pi i\; n(\gamma,a) \operatorname*{Res}_a f.$$

2. Let $g(z) = p(z)/q(z)$ be a rational function, such that $\deg q \geq 2 + \deg p$. Show that the sum of residues of g at all its singularities is zero.

3. Evaluate the following integrals:

(a) $\displaystyle\int_0^\pi \frac{d\theta}{4+\sin^2\theta}$;

(b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx$;

(c) $\displaystyle\int_{-\infty}^\infty \frac{\sin \mu x}{x(a^2+x^2)}\,dx$, where $a>0,\ \mu>0$;

(d) $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx.$

4. For $\alpha \in (-1,1)$ with $\alpha \neq 0$ compute

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1}\,dx.$$

5. Use Rouche's Theorem to prove the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then p has n roots (counted with multiplicity) in the disc $\{|z| < A+1\}$.

6. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re}\ p(z)$ for such z. Hence sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$, and use your sketch to determine the number of z (counted with multiplicity), such that $|z| < 1$ and $p(z) = x$ for each real value x.

7. (i) Show that $z^4 + 12z + 1$ has exactly three zeros in the annulus $\{z \in \mathbb{C} : 1 < |z| < 4\}$. Show that these zeros are distinct.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{z : |z| = 3/2\}$.

8. Suppose that f is holomorphic on some open disc containing $\{|z| \leq 1\}$ and satisfies $|f(z)| < 1$ when $|z| = 1$. Show that there is exactly one complex number w, such that $|w| < 1$ and $f(w) = w$.

9. (Inverse function formula for holomorphic functions.) Let f be an analytic function on a disc $D(a,R)$, such that $f'(a) \neq 0$. Show that for sufficiently small $r > 0$ the formula

$$g(w) = \frac{1}{2\pi i}\int_{|z-a|=r} z\,\frac{f'(z)}{f(z)-w}\,dz$$

defines a holomorphic function on some neighbourhood of $f(a)$ which is inverse to f.

10. Prove that the equation $z \sin z = 1$ has only real roots.

[Hint: find the number of real roots in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$ and compare with the number of zeros of $z \sin z - 1$ in the disc $\{|z| < (n+1/2)\pi\}$.]

11. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N+1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$ around γ_N, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1+\pi \coth \pi}{2} \ .$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.

The following is *not* part of the example sheet, but may be added to revision or used as optional extras for enthusiasts.

(A1) Evaluate:

(i) $\displaystyle\int_0^\infty \sin x^2 \, dx$;

(ii) $\displaystyle\int_0^\infty \frac{x^\alpha dx}{(x+a)(x+2a)}$, for $-1 < \alpha < 1$, $a > 0$;

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx} \, dx$, where $t \in \mathbb{R}$;

(iv) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} \, dx$ ('dog-bone' contour).

(A2) By integrating $\dfrac{z}{a - e^{-iz}}$ around the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^\pi \frac{x \sin x}{1 - 2a \cos x + a^2} \, dx = \frac{\pi}{a} \log(1+a) \,, \qquad \text{for } 0 < a < 1.$$

(A3) (i) Show that the Taylor expansion of $z/(e^z - 1)$ near the origin has the form

$$1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k}{(2k)!} z^{2k} \ ,$$

where the numbers B_k (the *Bernoulli numbers*) are rational.

(ii) If k is a positive integer show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1} \pi^{2k} B_k}{(2k)!} \ .$$