COMPLEX ANALYSIS EXAMPLES 2

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Use the Cauchy integral formula to compute $\int_{|z|=2} \frac{dz}{z^2+1}$ and $\int_{|z|=2} \frac{dz}{z^2-1}$. Are the answers an accident? Formulate and prove a result for a polynomial with n distinct roots.

2. (i) Use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz$$

where $\alpha \in \mathbb{C}$.
(ii) By considering suitable complex integrals, show that if $r \in (0,1)$,

$$\int_0^{\pi} \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2} \qquad \text{and} \qquad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta = 2\pi.$$

3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if any one of the following conditions hold, then f is constant:
(i) $f(z)/z \to 0$ as $|z| \to \infty$.
(ii) There exists $b \in \mathbb{C}$ and $\varepsilon > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - b| > \varepsilon$.
(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

4. Let $f\colon D(a,r) \to \mathbb{C}$ be holomorphic, and suppose that $z = a$ is a local maximum for $\mathrm{Re}(f)$. Show that f is constant.

5. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.
(ii) Show that an entire function f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

6. (i) (Schwarz's Lemma) Let f be analytic on $D(0,1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.
(ii) Use Schwarz's Lemma to prove that any conformal equivalence from $D(0,1)$ to itself is given by a Möbius transformation.

7. (i) Let f be an entire function such that for every positive integer n, $f(1/n) = 1/n$. Show that $f(z) = z$.
(ii) Let f be an entire function with $f(n) = n^2$ for every $n \in \mathbb{Z}$. Must $f(z) = z^2$?
(iii) Let f be holomorphic on $D(0,2)$. Show that for some integer $n > 0$, $f(1/n) \neq 1/(n+1)$.

8. (i) Give an example of an infinitely differentiable function $f : (-1,1) \to \mathbb{R}$ which can be extended to a holomorphic function on a domain $U \subset \mathbb{C}$ containing $(-1,1)$, but for which one cannot take U to be the open unit disc $D(0,1)$.
(ii) Give an example of an infinitely differentiable function $f : (-1,1) \to \mathbb{R}$ which is not the restriction of any holomorphic function defined on a domain $U \subset \mathbb{C}$ containing $(-1,1)$.
(iii) Prove that the integral $\int_0^\infty e^{-zt} \sin(t)\, dt$ converges for $\mathrm{Re}(z) > 0$ and defines a holomorphic function in that half-plane. Prove furthermore that the resulting holomorphic function admits an analytic continuation to $\mathbb{C} \setminus \{\pm i\}$.
(iv) Show that the power series $\sum_{n=1}^\infty z^{n!}$ defines an analytic function f on $D(0,1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0,1)$. [Hint: consider $z = \exp(2\pi i p/q)$ with p/q rational.]

9. (i) Let $w \in \mathbb{C}$, and let $\gamma,\ \delta\colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve

$$\sigma(t) = \frac{\delta(t) - w}{\gamma(t) - w}$$

about the origin, show that $I(\gamma; w) = I(\delta; w)$.
(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w,r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w,r)$.
(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

10. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \bigm| |z| > 2\}.$$

11. Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

12. (Casorati-Weierstrass theorem) Let f be holomorphic on $D(a,R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a,R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

13. Let f be a holomorphic function on $D(a,R) \setminus \{a\}$. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\mathrm{Re}\, f(z) < M$ for $z \in D(a,R) \setminus \{a\}$, then f has a removable singularity at $z = a$.