Complex Analysis IB: 2015-16 – Sheet 3

1. Let f be a meromorphic function on $\mathbb{C}$ for which $|f(z)| \to \infty$ as $|z| \to \infty$. Show that f cannot have poles at all integer points.

2. Let $g(z) = p(z)/q(z)$ be a rational function with $\deg(q) \geq \deg(p) + 2$. Show that the sum of the residues of g over all its singularities is zero.

3. Prove that the group of conformal automorphisms of the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ is the Möbius group. [*Hint: take an automorphism g fixing 0 and ∞ and consider $z \mapsto g(z)/z$.*]

4. Evaluate the following:

$$(a) \quad \int_0^\pi \frac{d\theta}{4 + \sin^2\theta}; \qquad (b) \quad \int_0^\infty \frac{x^2\, dx}{(x^2+4)^2(x^2+9)};$$

$$(c) \quad \int_0^\infty \sin x^2\, dx; \qquad (d) \quad \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\, dx.$$

5. For $-1 < \alpha < 1$ and $\alpha \neq 0$, compute

$$\int_0^\infty \frac{x^\alpha}{1 + x + x^2}\, dx.$$

6. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i|,\ 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $\{|z| < A + 1\}$.

7. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z. Sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$, and hence determine the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each $x \in \mathbb{R}$.

8. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

   (ii) By integrating $\dfrac{\pi \cot \pi z}{z^2 + 1}$ around γ_N, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi \coth\pi}{2}.$$

   (iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.

9. Show that the Taylor expansion of $z/(e^z - 1)$ near the origin has the form

$$1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k}{(2k)!} z^{2k}$$

   where the numbers B_k (the *Bernoulli numbers*) are rational.

10. Let $w \in \mathbb{C}$, and let $\gamma, \delta \colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number $I(\sigma; 0)$ about the origin for the closed curve $\sigma(t) = (\delta(t) - w)/(\gamma(t) - w)$, show that $I(\gamma; w) = I(\delta; w)$.

    (ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $B(w; r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in B(w; r)$. Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

11. (i) Show that $z^4 + 12z + 1$ has exactly three zeros in the annulus $\{1 < |z| < 4\}$.

    (ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \mathrm{Re}(z) < 0 \,\}$.

    (iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z \mid |z| = 3/2 \,\}$.

12. Show that the equation $z \sin z = 1$ has only real solutions.

    [*Hint: Find the number of real roots in the interval* $[-(n+1/2)\pi, (n+1/2)\pi]$ *and compare with the number of zeroes of* $z \sin z - 1$ *in a square box* $\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)| < (n+1/2)\pi\}$.]

13* (Additional) Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) \;=\; \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

The following integrals are *not* part of the question sheet, but may provide a good start for revision or a first port of call for the addicted.

(*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2 + x^2)}\, dx$ where $a,\ m \in \mathbb{R}^+$;

(*ii*) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1 - 2a\cos t + a^2}\, dt$ where $a \in (0,1)$;

(*iii*) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx$ ("dog-bone" contour);

(*iv*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx$, where $t \in \mathbb{R}$.

(*v*) By integrating $z/(a - e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log(1 + a) \quad \text{for } a \in (0,1).$$

(*vi*) Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that

$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$

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