

Complex Analysis IB: 2015-16 – Sheet 2

Let $B(a; \epsilon)$ denote the open ball $\{z \in \mathbb{C} : |z - a| < \epsilon\}$.

1. The Weierstrass approximation theorem states that any continuous function $f : I \rightarrow \mathbb{R}$ on a closed bounded connected subset $I \subset \mathbb{R}$ can be uniformly approximated by polynomials. Can any continuous function $\phi : J \rightarrow \mathbb{C}$ on a closed bounded connected subset $J \subset \mathbb{C}$ be uniformly approximated by polynomials? Justify your answer.

2. (i) Using the Cauchy integral formula, compute $\int_{|z|=2} \frac{1}{z^2+1} dz$ and $\int_{|z|=2} \frac{1}{z^2-1} dz$.

(ii) If $p(z)$ is a polynomial with distinct roots $\{a_j\}$, what is the maximum conceivable number of distinct values that $\int_{\gamma} \frac{1}{p(z)} dz$ can take, as γ varies over (piecewise C^1 -smooth) simple closed curves disjoint from the $\{a_j\}$? [You are not asked to provide a polynomial p for which this theoretical maximum is realised.]

3. (i) For $\alpha \in \mathbb{C}$, use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2} dz.$$

(ii) By considering suitable complex integrals, show that

$$\int_0^{\pi} \frac{\cos n\theta}{1 - 2r \cos \theta + r^2} d\theta = \frac{\pi r^n}{1 - r^2} \quad \forall r \in (0, 1); \quad \text{and} \quad \int_0^{2\pi} \cos(\cos \theta) \cosh(\sin \theta) d\theta = 2\pi.$$

4. Let f be an entire function.

- (i) If $f(z)/z \rightarrow 0$ as $|z| \rightarrow \infty$, prove that f is constant. (This strengthens Liouville's theorem.)
- (ii) If for some $a \in \mathbb{C}$ and $\epsilon > 0$, f never takes values in $B(a; \epsilon)$, show that f is constant.
- (iii) If $f = u + iv$ and $|u| > |v|$ throughout \mathbb{C} , show that f is constant.

5. Let U be a domain and $f : U \rightarrow \mathbb{C}$ be holomorphic. If the real part $\operatorname{Re}(f)$ has an interior local maximum at $a \in U$, show that f is constant.

6. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z .

(ii) Show that an entire function f is a polynomial if and only if $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.

(iii) Let f be a function which is holomorphic on \mathbb{C} apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

7. (i) (Schwarz's Lemma) Let f be holomorphic on the open unit disk $B(0; 1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c .

(ii) Use Schwarz's Lemma to prove that any conformal equivalence from the unit disk to itself is given by a Möbius transformation.

8. (i) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. If $f(1/n) = 1/n$ for each $n \in \mathbb{Z}_{>0}$, show that $f(z) = z$.
(ii) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. If $f(n) = n^2$ for every $n \in \mathbb{Z}$, must $f(z) = z^2$?
(iii) Let f be holomorphic on $B(0; 2)$. Show that $f(1/n) \neq 1/(n+1)$ for some $n \in \mathbb{Z}_{>0}$.
9. (i) Give an example of an infinitely differentiable function $f : (-1, 1) \rightarrow \mathbb{R}$ which can be extended to a holomorphic function on a domain $(-1, 1) \subset U \subset \mathbb{C}$, but for which one cannot take U to be the open unit disc $B(0; 1)$.
(ii) Give an example of an infinitely differentiable function $f : (-1, 1) \rightarrow \mathbb{R}$ which is not the restriction of any holomorphic function defined on a domain $(-1, 1) \subset U \subset \mathbb{C}$.
(iii) Prove that the integral $\int_0^\infty e^{-zt} \sin(t) dt$ converges for $\operatorname{Re}(z) > 0$ and defines a holomorphic function in that half-plane. Prove furthermore that the resulting holomorphic function admits an analytic continuation to $\mathbb{C} \setminus \{\pm i\}$.
(iv) Prove that the series $\sum_{n=0}^\infty z^{(2^n)}$ defines a holomorphic function on the disc $B(0; 1)$ which admits no analytic continuation to any larger domain $B(0; 1) \subsetneq U \subset \mathbb{C}$.

10. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \mid |z| > 2\}.$$

11. Classify the singularities of each of the following functions:

$$\frac{1}{z^2} + \frac{1}{z^2 + 1}, \quad \frac{z}{\sin z}, \quad \sin \frac{\pi}{z^2}, \quad \frac{1}{z^2} \cos \left(\frac{\pi z}{z+1} \right).$$

12. (Casorati-Weierstrass theorem) Let f be holomorphic on $B(a; r) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in B(a; r)$ with $z_n \neq a$ such that $z_n \rightarrow a$ and $f(z_n) \rightarrow b$ as $n \rightarrow \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, a holomorphic function takes *every* complex value except possibly one.]

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