Complex Analysis IB: 2015-16 – Sheet 1

1. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that T can be written $Tz = Az + B\bar{z}$ for unique $A, B \in \mathbb{C}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a domain (non-empty path-connected open subset) D. Show that f is constant if any of its real part, modulus or argument is constant.

   (ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x+iy) = u(x) + iv(y)$ where u and v are both real valued.

   (iii) Find all holomorphic functions $f(z)$ on $\mathbb{C}$ which have real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and
$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2+y^2} \qquad \text{for } z = x+iy \neq 0.$$
   Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4. (i) Verify directly that e^z and $\cos z$ satisfy the Cauchy-Riemann equations everywhere.

   (ii) Find the set of $z \in \mathbb{C}$ for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

   (iii) Find the zeros of $1 + e^z$ and $\cosh z$.

5. (i) Defining $z^\alpha = e^{\alpha \operatorname{Log} z}$, for Log the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz} z^\alpha = \alpha\, z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold ?

   (ii) If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce
$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad \forall\, z \in \mathbb{C}.$$

6. Find conformal equivalences between the following pairs of domains:

   (i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc D;

   (ii) the lune $\{z \in \mathbb{C} : |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$ and the open unit disk D;

   (iii) the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.

   By considering a suitable bounded solution of the Laplace equation $u_{xx} + u_{yy} = 0$ on the strip S, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant (it need not be continuous at the origin).

7. (i) Show that the general Möbius transformation which takes the unit disk to itself has the form $z \mapsto \lambda \frac{z-a}{\bar{a}z-1}$, with $|a| < 1$, $|\lambda| = 1$. [*Hint: first show these maps form a group.*]

   (ii) Find a Möbius transformation taking the region between $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: A circle can be described by an equation of the shape* $|z-a|/|z-b| = l$.]

   (iii) Find a conformal map from an infinite strip onto an annulus. Can such a map be the restriction to the strip of a Möbius map ?

8. Prove that the following series converge uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n}e^{-nz}, \quad \text{on } \{0 < \mathrm{Re}(z)\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \quad \text{on } \left\{|z| < \frac{1}{2}\right\}.$$

9. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

10. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivative) on the domains indicated:

$$(a)\ \frac{1}{z} - \frac{1}{z-1} \qquad (0 < |z| < 1); \qquad (b)\ \frac{z}{1+z^2} \qquad (1 < |z| < \infty).$$

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