Complex Analysis IB – 2012 – Sheet 2

Recall that $B_a(\epsilon)$ denotes the open ball $\{z \in \mathbb{C} : |z - a| < \epsilon\}$.

1. (i) Using the Cauchy integral formula, compute $\int_{|z|=2} \frac{dz}{z^2+1}$ and $\int_{|z|=2} \frac{dz}{z^2-1}$.

   (ii) If $p(z)$ is a polynomial with distinct roots $\{a_j\}$, how many distinct values can $\int_\gamma \frac{dz}{p(z)}$ take, as γ varies over simple closed curves disjoint from the $\{a_j\}$?

2. (i) For $\alpha \in \mathbb{C}$, use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2} \, dz.$$

   (ii) By considering suitable complex integrals, show that

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r\cos\theta + r^2} \, d\theta = \frac{\pi r^n}{1 - r^2} \ \forall\, r \in (0,1); \quad \text{and} \quad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta \ = \ 2\pi.$$

3. Let f be an entire function.

   (i) If $f(z)/z \to 0$ as $|z| \to \infty$, prove that f is constant. (This strengthen's Liouville's theorem.)

   (ii) If for some $a \in \mathbb{C}$ and $\epsilon > 0$, f never takes values in $B_a(\epsilon)$, show that f is constant.

   (iii) If $f = u + iv$ and $|u| > |v|$ throughout $\mathbb{C}$, show that f is constant.

   (iv) By considering

$$\phi : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}\backslash[-1,1] \quad z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$$

   show that if f never takes values in the segment $[-1,1] \subset \mathbb{R}$, then f is constant.

4. Let U be a domain and $f : U \to \mathbb{C}$ be holomorphic. If the real part $\Re(f)$ has an interior local maximum at $a \in U$, show that f is constant.

5. (i) Let f be an entire function such that for every positive integer n one has $f(1/n) = 1/n$. Show that $f(z) = z$.

   (ii) Let f be holomorphic on $B_0(2)$. Show that $f(1/n) \neq 1/(n+1)$ for some $n \in \mathbb{Z}_{>0}$.

   (iii) Show that there is no holomorphic function $f\colon B_0(1) \to \mathbb{C}$ such that $f(z)^2 = z$.

   (iv) Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(n) = n^2$ for every $n \in \mathbb{Z}$, does it follow that $f(z) = z^2$?

6. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.

   (ii) Show that an entire function f is a polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

   (iii) Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

   (iv) Let f be a meromorphic function on $\mathbb{C}$ such that $f(1/z)$ is also meromorphic on $\mathbb{C}$. Show that f is a rational function.

7. (i) (Schwarz's Lemma) Let f be analytic on the open unit disk D, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.

   (ii) Use Schwarz's Lemma to prove that any conformal equivalence from the unit disk to itself is given by a Möbius transformation.

8. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \mid |z| > 2\}.$$

   Also find its Laurent expansion (in powers of $z - 1$) in the region $\{z \mid 0 < |z - 1| < 1\}$.

9. Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

10. Let f have an isolated singularity at $z = a$ which is not an essential singularity. If f is not identically zero, show that there exists $r > 0$ such that $f(z) \neq 0$ whenever $0 < |z - a| < r$.

11. (Casorati-Weierstrass theorem) Let f be holomorphic on $B_a(R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in B_a(R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

    Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

    [A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

12. Let $f\colon B_a(R) \setminus \{a\} \to \mathbb{C}$ be holomorphic. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\Re f(z) < M$ for $z \in B_a(R)$, then f has a removable singularity at $z = a$.

13. (i) Let $w \in \mathbb{C}$, and let $\gamma, \delta\colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number $n(\sigma; 0)$ of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $n(\gamma; w) = n(\delta; w)$.

    (ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $B_w(r)$, show that $n(\gamma; w) = n(\gamma; z)$ for every $z \in B_w(r)$.

    (iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto n(\gamma; w)$ is a locally constant function on U.

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