Complex Analysis IB – 2012 – Sheet 1

1. Let $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that T can be written $Tz = Az + B\bar{z}$ for unique $A, B \in \mathbb{C}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a domain (path-connected open subset) D. Show that f is constant if any of its real part, modulus or argument is constant.

   (ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

   (iii) Find all holomorphic functions $f(z)$ on $\mathbb{C}$ which have real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

   Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4. (i) Verify directly that e^z, $\cos z$ and $\sin z$ satisfy the Cauchy-Riemann equations everywhere.

   (ii) Find the set of complex numbers z for which $|e^z| < 1$, the set of those for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

   (iii) Find the zeros of $1 + e^z$, $\cosh z$ and $\sin z + \cos z$.

5. (i) Defining $z^\alpha = e^{\alpha \operatorname{Log} z}$, for Log the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz} z^\alpha = \alpha\, z^{\alpha - 1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold ?

   (ii) If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad \forall\, z \in \mathbb{C}.$$

   (iii) Find the radius of convergence R of the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. Determine whether or not the series converges on the circle $|z| = R$.

6. Find conformal equivalences between the following pairs of domains:

   (i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc D;

   (ii) the lune $\{z \in \mathbb{C} : |z - 1| < \sqrt{2} \text{ and } |z + 1| < \sqrt{2}\}$ and the open unit disk D;

   (iii) the strip $S = \{z \in \mathbb{C} : 0 < \Im z < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \Re z > 0, \Im z > 0\}$.

   By considering a suitable bounded solution of the Laplace equation $u_{xx} + u_{yy} = 0$ on the strip S, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant.

7. (i) Find all the Möbius transformations which take the unit disk to itself.

   (ii) Find a Möbius transformation taking the region between $\{|z| = 1\}$ and $\{|z - 1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. Is there any choice in the value of R ?

   (iii) Find a conformal map from an infinite strip onto an annulus. Can such a map be the restriction to the strip of a Möbius map ?

8. (*Hadamard's formula*) Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ is given by

$$R = \frac{1}{\limsup \sqrt[n]{|c_n|}}.$$

[*Recall that if* $\{x_n\}$ *is a sequence of real numbers, and* $M_n = \sup_{r \geq n} x_r$, $m_n = \inf_{r \geq n} x_r$ *then*

$$\limsup\{x_n\} = \lim_{n\to\infty} M_n, \quad \liminf\{x_n\} = \lim_{n\to\infty} m_n, \qquad \textit{with } \pm\infty \textit{ allowed.}]$$

9. Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n} e^{-nz}, \quad \text{on } \{\, z \mid 0 < r \leq \Re(z) \,\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \quad \text{on } \{\, z \mid |z| \leq r < \frac{1}{2} \}.$$

10. Evaluate the integrals

$$\int_\gamma |z|^2\, dz, \qquad \int_\gamma z^2\, dz$$

when $\gamma : [0,1] \to \mathbb{C}$ is given by (a) $\gamma(t) = e^{i\pi t/2}$, and (b) $\gamma(t) = 1 - t + it$.

11. (Integration by parts) Let f and g be holomorphic in a domain D, and let $\gamma\colon [0,1] \to D$ be a curve with $\gamma(0) = a$, $\gamma(1) = b$. Show that

$$\int_\gamma f(z) g'(z)\, dz = f(b)g(b) - f(a)g(a) - \int_\gamma f'(z) g(z)\, dz.$$

12. Calculate $\int_\gamma z \sin z\, dz$ when γ is the straight line joining 0 to i.

13. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivative) on the domains indicated:

$$(a)\quad \frac{1}{z} - \frac{1}{z-1} \qquad (0 < |z| < 1); \qquad (b)\quad \frac{z}{1+z^2} \qquad (1 < |z| < \infty).$$

14. (i) Let $D \subset \mathbb{C}$ be a domain not containing 0, and $\lambda\colon D \to \mathbb{C}$ a branch of the logarithm. Determine all possible branches of the logarithm on D in terms of λ.

(ii) If $l \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $w \in \mathbb{C}$ satisfies $e^w = l$, find a continuously differentiable curve $\gamma : [0,1] \to \mathbb{C}^*$ from 1 to l with the property that $\int_\gamma \frac{1}{z} dz = w$. What does this say about the logarithm?

(iii) If $H : \mathbb{C} \to \mathbb{C}^*$ is entire and has no zeroes, show there is an analytic function $h : \mathbb{C} \to \mathbb{C}$ for which $H(z) = e^{h(z)}$ for every z.

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