Part IB COMPLEX ANALYSIS (Lent 2009): Example Sheet 3

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Comments and/or corrections are welcome at any time and can be emailed to me at `a.g.kovalev@dpmms.cam.ac.uk`. This sheet is for most part based on the questions given by Prof. Scholl last year, though I made some modifications.

1. (i) Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$ and $|w-a|<r<R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}}\,dz$$

(ii) Let $g(z)=p(z)/q(z)$ be a rational function, such that $\deg q \geq 2+\deg p$. Show that the sum of residues of g at all its singularities is zero.

2. Evaluate:

(a) $\displaystyle\int_0^\pi \frac{d\theta}{4+\sin^2\theta}\;;$

(b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx\,;$

(c) $\displaystyle\int_{-\infty}^\infty \frac{\sin\mu x}{x(a^2+x^2)}\,dx\,,$ where $a>0,\ \mu>0\,;$

(d) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1-2a\cos t+a^2}\,dt\,,$ where $0<a<1$.

3. Use Rouche's Theorem to give another proof of the Fundamental Theorem of Algebra.

4. (i) Show that $z^4+12z+1$ has exactly three zeros in the annulus $\{z\in\mathbb{C}: 1<|z|<4\}$. Show that these zeros are distinct.

(ii) Prove that z^5+2+e^z has exactly three zeros in the half-plane $\{z\in\mathbb{C}: \operatorname{Re} z<0\}$.

5. Let $p(z)=z^5+z$. Find all z such that $|z|=1$ and $\operatorname{Im} p(z)=0$. Calculate $\operatorname{Re}\ p(z)$ for such z. Hence sketch the curve $p\circ\gamma$, where $\gamma(t)=e^{2\pi i t}$, and use your sketch to determine the number of z (counted with multiplicity), such that $|z|<1$ and $p(z)=x$ for each real value x.

6. (Jensen's formula.) Let f be a holomorphic function on a domain containing a closed disc $\overline{B}=\{|z|\leq\rho\}$. Show that if f has no zeros in $\overline{B}$ then f'/f has an antiderivative on some open set containing $\overline{B}$. Deduce from the Cauchy Integral Formula that

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta}|d\theta$$

Now suppose that $w_1,\ldots,w_n$ are all the zeros of f in the open disc $B=\{|z|<\rho\}$, repeated according to multiplicities, but f never vanishes on the boundary $\{|z|=\rho\}$ and $f(0)\neq 0$. Show that then

$$\log|f(0)| = -\sum_{k=1}^n \log\left(\frac{\rho}{|w_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta}|d\theta$$

[Hint: consider the function $f(z)\prod_{k=1}^n\big((\rho^2-\bar{w}_k z)/(\rho(z-w_k))\big)$.]

7. (Inverse function formula for holomorphic functions.) Let f be an analytic function on a disc $D(a,R)$, such that $f'(a)\neq 0$. Show that for sufficiently small r the formula

$$g(w) = \frac{1}{2\pi i}\int_{|z-a|=r} z\,\frac{f'(z)}{f(z)-w}\,dz$$

defines a holomorphic function on some neighbourhood of $f(a)$ which is inverse to f.

8. Evaluate:

(a) $\displaystyle\int_0^\infty \frac{x^\alpha\,dx}{(x+a)(x+2a)}$, for $-1<\alpha<1$, $a>0$;

(b) $\displaystyle\int_0^\infty \sin x^2\,dx$ [substitute $u=x^2$];

9. By integrating $\dfrac{z}{a-e^{-iz}}$ around the rectangle with vertices $\pm\pi$, $\pm\pi+iR$, prove that

$$\int_0^\pi \frac{x\sin x}{1-2a\cos x+a^2}\,dx = \frac{\pi}{a}\log(1+a)\,, \qquad \text{for } 0<a<1.$$

10. Evaluate:

(a) $\displaystyle\int_{-\infty}^\infty e^{-ax^2}e^{-itx}\,dx$, where $a>0$, $t\in\mathbb{R}$ [you may assume that $\displaystyle\int_{-\infty}^\infty e^{-x^2/2}dx=\sqrt{2\pi}$, use a rectangular contour with one side on the real axis];

(b) $\displaystyle\int_0^\infty \frac{\log(x^2+1)}{x^2+1}\,dx$ [use an upper semicircle];

(c) $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x}\,e^{-itx}\,dx$, where $t\in\mathbb{R}$.

11. Show that the equation $z^4+z+1=0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{|z|=3/2\}$.

12. Suppose that f is holomorphic on some open disc containing $\{|z|\le 1\}$ and satisfies $|f(z)|<1$ when $|z|=1$. Show that there is exactly one complex number w, such that $|w|<1$ and $f(w)=w$.

13. Prove that the equation $z\sin z=1$ has only real roots.

[Hint: find the number of real roots in the interval $[-(n+1/2)\pi,(n+1/2)\pi]$ and compare with the number of zeros of $z\sin z-1$ in the disc $\{|z|<(n+1/2)\pi\}$.]

14. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm1\pm i)(N+1/2)$. Show that there exists $C>0$ such that for every N, $|\cot\pi z|<C$ on γ_N.

(ii) By integrating $\dfrac{\pi\cot\pi z}{z^2+1}$ around γ_N, show that

$$\sum_{n=0}^\infty \frac{1}{n^2+1} = \frac{1+\pi\coth\pi}{2}\,.$$

(iii) Evaluate $\sum_{n=0}^\infty(-1)^n/(n^2+1)$.

15. (i) Show that the Taylor expansion of $z/(e^z-1)$ near the origin has the form

$$1-\frac{z}{2}+\sum_{k=1}^\infty \frac{(-1)^{k-1}B_k}{(2k)!}z^{2k}\,,$$

where the numbers B_k (the *Bernoulli numbers*) are rational.

(ii) If k is a positive integer show that

$$\sum_{n=1}^\infty \frac{1}{n^{2k}} = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!}\,.$$