Part IB COMPLEX ANALYSIS (Lent 2009): Example Sheet 2

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Comments and/or corrections are welcome at any time and can be emailed to me at `a.g.kovalev@dpmms.cam.ac.uk`. This sheet is for most part based on the questions given by Prof. Scholl last year, though I made some modifications.

1. (i) Use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{3z^2 - 7z + 2} dz \,,$$

where $\alpha \in \mathbb{C}$.

(ii) By considering the real part of a suitable complex integral, show that for all $r \in (0, 1)$,

$$\int_0^\pi \frac{\cos(n\theta)}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2}.$$

2. Strengthen Liouville's theorem by showing that if f is an entire function such that $f(z)/z \to 0$ as $|z| \to \infty$, then f is constant.

3. Let f be an entire function which, for some $a \in \mathbb{C}$ and $\varepsilon > 0$, never takes values in the disc $D(a, \varepsilon)$. Prove that f is constant.

4. Show that

$$\varphi : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C} \setminus [-1, 1], \quad z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a conformal map between the two domains. If an entire function f never takes values in the line segment $[-1, 1]$, show that $\varphi^{-1} \circ f$ is holomorphic and deduce that f must be constant.

5. Let f be an analytic function on a disc $D(w, R)$. Show that for every $r < R$,

$$\left|f^{(n)}(w)\right| \le \frac{n!}{r^n} \sup_{|z-w|=r} |f(z)|.$$

6. (i) Let f be an entire function such that for every positive integer n one has $f(1/n) = 1/n$. Show that $f(z) = z$.

(ii) Let h be a holomorphic function on the disc $\{z \in \mathbb{C} : |z| < 2\}$. Show that there exists a positive integer n such that $h(1/n) \neq 1/(n+1)$.

7. Show that there is no holomorphic function $f : D(0, 1) \to \mathbb{C}$ such that $f(z)^2 = z$.

8. Find the Laurent expansion, in powers of z, of $1/(z^2 - 3z + 2)$ in each of the domains:

$$\{z \in \mathbb{C} : |z| < 1\}, \qquad \{z \in \mathbb{C} : 1 < |z| < 2\}, \qquad \{z \in \mathbb{C} : |z| > 2\}.$$

Also find its Laurent expansion, in powers of $z - 1$, in the domain $\{z \in \mathbb{C} : 0 < |z - 1| < 1\}$.

9. Classify the singularities of each of the holomorphic functions:

$$\frac{z}{\sin z}, \qquad \frac{1}{z^4 + z^2}, \qquad \cos\frac{\pi}{z^2}, \qquad \frac{1}{z^2}\cos\frac{\pi z}{z + 1}.$$

10. (Casorati–Weierstrass theorem) Let f be holomorphic on a punctured disc $D^*(a,r)$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a,r)$, with $z_n \neq a$, such that $z_n \to a$ and $f(z_n) \to b$, as $n \to \infty$.

[Hint: you might like to consider a function $g(z) = \dfrac{1}{f(z) - b}$.]

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard asserts that in any neighbourhood of an essential singularity a holomorphic function takes *every* complex value except possibly one.]

11. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.

(ii) Show that an entire function is a polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

12. Let f be a function which is holomorphic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists $k \in \mathbb{Z}$ such that $|f(z)| < |z|^k$, for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

13. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(n) = n^2$, for every $n \in \mathbb{Z}$, does it follow that $f(z) = z^2$?

14. (i) Let $w \in \mathbb{C}$ and let $\gamma, \delta : [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number $n(\sigma, 0)$ of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $n(\gamma, w) = n(\delta, w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$ and γ is a closed curve which does not meet $D(w,r)$, show that $n(\gamma, w) = n(\gamma, z)$ for every $z \in D(w,r)$.

(iii) Deduce that if γ is a closed curve and U is the complement of (the image of) γ then the function $w \mapsto n(\gamma, w)$ is a locally constant function on U.

15. Let f be a meromorphic function on $\mathbb{C}$ such that $f(1/z)$ is also meromorphic. Show that f is a rational function.

16. Let f be a holomorphic function on a punctured disc $D^*(a,R)$. Show that if f has a non-removable singularity at $z = a$ then the function $\exp(f(z))$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\operatorname{Re} f(z) < M$ for $z \in D^*(a,R)$, then f has a removable singularity at $z = a$.