Part IB COMPLEX ANALYSIS (Lent 2009): Example Sheet 1

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Comments and/or corrections are welcome at any time and can be emailed to me at `a.g.kovalev@dpmms.cam.ac.uk`. This sheet is based on the questions given by Prof. Scholl last year, though I made some modifications.

1. Show that any real linear map $T : \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{C} \simeq \mathbb{R}^2$ can be written as $T(z) = Az + B\bar{z}$, for two complex numbers A and B. Considering T as a complex-valued function on $\mathbb{C}$, deduce that T is complex differentiable on $\mathbb{C}$ if and only if $B = 0$.

2. Show that the function $f(z) = z\bar{z}$ is complex differentiable as $z = 0$ and nowhere else in $\mathbb{C}$. Show that $|z|$ is nowhere complex differentiable.

3. (i) Let $f : D(a, r) \to \mathbb{C}$ be a holomorphic function on a disc. Show that f is constant if either its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$, where u and v are both real valued.

(iii) Find all the functions which are holomorphic on $\mathbb{C}$ and which have the real part $x^3 - 3xy^2$. (The functions that you find should be given in terms of the complex variable z.)

4. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$ and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies Cauchy–Riemann equations at 0 but is not differentiable there.

5. Let $f : D(a, r) \to \mathbb{C}$ be a complex differentiable function on a disc about $a \in \mathbb{C}$, with $f'(a) = b$, and define $\varphi(z) = \overline{f(\bar{z})}$. Show that φ is complex differentiable on $D(\bar{a}, r)$ and that $\varphi'(\bar{a}) = \bar{b}$.

6. Find the radius of convergence R of the series $\sum_{n=1}^{\infty} \dfrac{n!}{n^n} z^n$. Determine whether or not the series converges on the circle $|z| = R$.

7. (Hadamard's formula) Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ is given by

$$R = \frac{1}{\limsup \sqrt[n]{|c_n|}}.$$

[Recall that is $\{x_n\}$ is a sequence of real numbers, and $M_n = \sup_{r \geq n} x_r$, $m_n = \inf_{r \geq n} x_r$ then

$$\limsup x_n = \lim_{n\to\infty} M_n, \qquad \liminf x_n = \lim_{n\to\infty} m_n$$

(with $\pm\infty$ allowed throughout).]

8. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with radius of convergence $R > 0$. Show (without using any form of Taylor's theorem) that if $|w| = R - r < R$ then $f(z)$ can be represented by a convergent power series $f(z) = \sum_{n=0}^{\infty} d_n (z - w)^n$ on the disc $D(w, r)$. What can you say about its radius of convergence?

9. Verify directly that e^z, $\cos z$ and $\sin z$ satisfy Cauchy–Riemann equations everywhere.

10. (i) Find the set of complex numbers z for which $|e^z| < 1$, the set of those for which $|e^{iz}| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

(ii) Find the zeros of $1 + e^z$, $\cosh z$, $\sinh z$ and $\sin z + \cos z$.

11. Let $D \subset \mathbb{C}$ be a domain not containing 0, and $\lambda : D \to \mathbb{C}$ a branch of the logarithm. Determine all possible branches of the logarithm on D in terms of λ.

12. Suppose that $f : D(-1, \varepsilon) \to \mathbb{C}$ is holomorphic and satisfies $\big(f(z)\big)^2 = z$ on $D(-1, \varepsilon)$, for some $\varepsilon > 0$, and $f(-1) = -i$. Show that $f'(-1) = i/2$.

13. Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 z^{2n}}$, on $\{z : |z| \geq 1\}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}\, e^{-nz}$, on $\{z : 0 < r \leq \operatorname{Re} z\}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}$, on $\{z : |z| \leq r < \frac{1}{2}\}$

(d) $\displaystyle\sum_{n=1}^{\infty} 2^{-n} \cos nz$, on $\{z : |\operatorname{Im} z| \leq r < \log 2\}$

14. Find the image of a sector $\{z \in \mathbb{C} : \alpha < \arg z < \beta\}$ (where $0 \leq \alpha < \beta \leq \pi$) under the Möbius transformation

$$z \mapsto \frac{z + i}{z - i}$$

15. Find conformal equivalences between the following pairs of domains:

(a) the sector $\{z \in \mathbb{C} : -\pi/3 < \arg z < \pi/3\}$ the open unit disc $D(0, 1)$;

(b) the horizontal strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ and the quadrant $\{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$;

(c) the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and the half-disc $\{z \in D(0, 1) : \operatorname{Re} z > 0\}$.

16. (Integration by parts.) Let f and g be holomorphic functions on a domain D and $\gamma : [0, 1] \to D$ a curve with $\gamma(0) = a$, $\gamma(1) = b$. Show that

$$\int_\gamma f(z) g'(z)\, dz = f(b)g(b) - f(a)g(a) - \int_\gamma f'(z) g(z)\, dz.$$

Calculate $\int_\gamma z \sin z\, dz$ when γ is the straight line joining 0 to i.

17. Show that the following functions do not have antiderivatives on the domains indicated:

(a) $\dfrac{1}{z}$ $\quad (0 < |z| < \infty)$; $\qquad$ (b) $\dfrac{z}{1 + z^2}$ $\quad (1 < |z| < \infty)$.

18. For each $\ell \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, such that $\exp(w) = \ell$, find a continuously differentiable curve $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ joining 1 and ℓ, such that $\int_\gamma \frac{1}{z}\, dz = w$.