1 Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$, and $|w-a|<r<R$, then

$$f^{(n)}(w)=\frac{n!}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{(z-w)^{n+1}}\,dz.$$

2 Show that if f and g are holomorphic and non-constant, then $\deg_{z=a}(g\circ f)=\deg_{z=a}(f).\deg_{z=f(a)}(g)$.

3 Use Rouché's Theorem to give another proof of the Fundamental Theorem of Algebra.

4 Let $p(z)=z^5+z$. Find all z such that $|z|=1$ and $\operatorname{Im} p(z)=0$. Calculate $\operatorname{Re} p(z)$ for such z. Hence sketch the curve $p\circ\gamma$, where $\gamma(t)=e^{2\pi it}$ and use your sketch to determine the number of z (counted with multiplicity) such that $|z|<1$ and $p(z)=x$ for each real number x.

5 Evaluate:

(a) $\displaystyle\int_0^\pi \frac{d\theta}{4+\sin^2\theta}$;

(b) $\displaystyle\int_{-\infty}^\infty \frac{\sin mx}{x(a^2+x^2)}\,dx$ where $a,\ m\in\mathbb{R}^+$;

(c) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx$;

(d) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1-2a\cos t+a^2}\,dt$ where $a\in(0,1)$.

6 By integrating $z/(a-e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi+iR$, prove that

$$\int_0^\pi \frac{x\sin x}{1-2a\cos x+a^2}\,dx=\frac{\pi}{a}\log(1+a)\quad\text{for } a\in(0,1).$$

7 Evaluate:

(a) $\displaystyle\int_0^\infty \sin x^2\,dx$

(b) $\displaystyle\int_{-\infty}^\infty e^{-ax^2}e^{-itx}\,dx$ where $a>0$, $t\in\mathbb{R}$

(c) $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$

(d) $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx$, $a\in(-1,1)$

(e) $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x}e^{-itx}\,dx$, $t\in\mathbb{R}$.

8 Assuming $\alpha\geq 0$ and $\beta\geq 0$ prove that

$$\int_0^\infty \frac{\cos\alpha x-\cos\beta x}{x^2}\,dx=\frac{\pi}{2}(\beta-\alpha),$$

and deduce the value of

$$\int_0^\infty\left(\frac{\sin x}{x}\right)^2 dx.$$

9 (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2 + 1}$ around γ_N, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi \coth\pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.

10 (i) Show that the Taylor expansion of $z/(e^z - 1)$ near the origin has the form

$$1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_n}{(2k)!} z^{2k}$$

where the numbers B_k (the *Bernoulli numbers*) are rational.

(ii) If k is a positive integer show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1} \pi^{2k} B_k}{(2k)!}.$$

11 Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \mathrm{Re}(z) < 0 \,\}$.

12 Show that $z^4 + 26z + 2 = 0$ has exactly three zeroes with $5/2 < |z| < 3$.

13 Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z \mid |z| = 3/2 \,\}$.

14 Show that the equation $z \sin z = 1$ has only real solutions.

15 Suppose f is holomorphic when $|z| \le 1$ and satisfies $|f(z)| < 1$ when $|z| = 1$. Show there is exactly one complex number w such that $|w| < 1$ and $f(w) = w$.

16 Let f be a meromorphic function on $\mathbb{C}$ such that $|f(z)| \to \infty$ as $|z| \to \infty$. Show that f cannot have poles at all integer points.

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