1 (i) Use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz$$

where $\alpha \in \mathbb{C}$.

(ii) By considering the real part of a suitable complex integral, show that for all $r \in (0,1)$,

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2}$$

2 Strengthen Liouville's theorem by showing that f is an entire function such that $f(z)/z \to 0$ as $|z| \to \infty$, then f is constant.

3 Let f be an entire function which, for some $a \in \mathbb{C}$ and $\epsilon > 0$, never takes values in $D(a, \epsilon)$. Prove that f is constant.

4 Let f be analytic on $D(w, R)$. Show that for every $r < R$,

$$\left| f^{(n)}(w) \right| \le \frac{n!}{r^n} \sup_{|z-w|=r} |f(z)|.$$

5 Let f be an entire function such that for every positive integer n one has $f(1/n) = 1/n$. Show that $f(z) = z$.

6 Show that there is no holomorphic function $f\colon D(0,1) \to \mathbb{C}$ such that $f(z)^2 = z$.

7 Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z \mid |z| < 1\}; \quad \{z \mid 1 < |z| < 2\}; \quad \{z \mid |z| > 2\}.$$

Also find its Laurent expansion (in powers of $z - 1$) in the region $\{z \mid 0 < |z - 1| < 1\}$.

8 Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

9 Let f have an isolated singularity at $z = a$ which is not an essential singularity. If f is not identically zero, show that there exists $r > 0$ such that $f(z) \neq 0$ whenever $0 < |z - a| < r$.

10 (Casorati-Weierstrass theorem) Let f be holomorphic on $D(a, R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a, R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

11 (i) Let f be an entire function. Show that f is a polynomial, of degree $\le k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.

(ii) Show that an entire function f is a polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

12 Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \le |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

13 Let $D \subset \mathbb{C}$ be a simply-connected domain which does not contain 0. Show that there exists a branch of the logarithm on D.

14 Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If $f(n) = n^2$ for every $n \in \mathbb{Z}$, does it follow that $f(z) = z^2$?

15 (i) Let $w \in \mathbb{C}$, and let γ, $\delta\colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $I(\gamma; w) = I(\delta; w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.

(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

16 Let f be a meromorphic function on $\mathbb{C}$ such that $f(1/z)$ is also meromorphic on $\mathbb{C}$. Show that f is a rational function.

17 (Schwarz's Lemma) Let f be analytic on $D(0, 1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.

18 Use Schwarz's Lemma to prove that any conformal equivalence from $D(0, 1)$ to itself is given by a Möbius transformation.

19 Let $f\colon D(a, R) \setminus \{a\} \to \mathbb{C}$ be holomorphic. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\operatorname{Re} f(z) < M$ for $z \in D(a, R)$, then f has a removable singularity at $z = a$.

20 Show that the power series $\sum_{n=1}^{\infty} z^{n!}$ defines an analytic function f on $D(0, 1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0, 1)$. [Hint: any such domain must contain a point $e^{2\pi i p/q}$ with $p/q \in \mathbb{Q}$.]

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