1 (i) Show that $f(z) = z\bar{z}$ is complex differentiable at the origin and nowhere else.

(ii) Show that the functions $|z|$, $\operatorname{Arg} z$, $\bar{z}$ are nowhere holomorphic.

2 Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

3 Let $f : D \to \mathbb{C}$ be an holomorphic function defined on a domain D. Show that f is constant if either its real part, imaginary part, modulus or argument is constant.

4 Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

5 Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2} & \text{when } z = x + iy \neq 0; \\ 0 & \text{when } z = 0. \end{cases}$$

Prove that $(f(z) - f(0))/z$ tends to 0 as $z \to 0$ along any straight line through the origin, but that f is not differentiable at the origin.

6 Find the radius of convergence R of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. Determine whether or not the series converges on the circle $|z| = R$.

7 (*Hadamard's formula*) Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ is given by

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

[Recall that if $\{x_n\}$ be a sequence of real numbers, and $M_n = \sup_{r \geq n} x_r$, $m_n = \inf_{r \geq n} x_r$ then

$$\limsup\{x_n\} = \lim_{n\to\infty} M_n, \quad \liminf\{x_n\} = \lim_{n\to\infty} m_n.$$

(with $\pm\infty$ allowed throughout).]

8 Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with radius of convergence $R > 0$. Show (without using any form of Taylor's theorem) that if $|w| = R - r < R$ then $f(z)$ can be represented by a convergent power series $f(z) = \sum_{n=0}^{\infty} d_n (z - w)^n$ on the disc $D(w, r)$. What can you say about its radius of convergence?

9 Verify directly that e^z, $\cos z$ and $\sin z$ satisfy the Cauchy-Riemann equations everywhere.

10 (i) Find the set of complex numbers z for which $|e^z| < 1$, the set of those for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

(ii) Find the zeros of $1 + e^z$, $\cosh z$, $\sinh z$ and $\sin z + \cos z$.

11 Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z.$$

12 Let $D \subset \mathbb{C}$ be a domain not containing 0, and $\lambda \colon D \to \mathbb{C}$ a branch of the logarithm. Determine all possible branches of the logarithm on D in terms of λ.

13 (i) Let $0 < r < 1 < R$. Show that the series

$$\sum_{k=1}^{\infty} \frac{z^k}{1+z^{2k}} \tag{1}$$

converges uniformly in each of the sets $D_r = \{z \in \mathbb{C} \mid |z| \leq r\}$ and $E_R = \{z \in \mathbb{C} \mid |z| \geq R\}$.

(ii) Show that the series (1) does *not* converge uniformly in either of the sets $D = \{z \in \mathbb{C} \mid |z| < 1\}$ or $E = \{z \in \mathbb{C} \mid |z| > 1\}$.

(iii) Show that on each of D and E the series (1) converges to a holomorphic function.

14 Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{n^2 z^{2n}}, \quad \text{on } \{\, z \bigm| |z| \geq 1 \,\}; \qquad (b)\ \sum_{n=1}^{\infty} \sqrt{n} e^{-nz}, \quad \text{on } \{\, z \bigm| 0 < r \leq \mathrm{Re}(z) \,\};$$

$$(c)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}, \quad \text{on } \{\, z \bigm| |z| \leq r < \frac{1}{2}\}; \qquad (d)\ \sum_{n=1}^{\infty} 2^{-n} \cos nz, \quad \text{on } \{\, z \bigm| |\mathrm{Im}(z)| \leq r < \log 2\}.$$

15 (i) Find a conformal equivalences between the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0,1)$.

(ii) Find the image of the sector $\{z \in \mathbb{C} \mid \alpha < \arg(z) < \beta\}$ (where $0 \leq \alpha < \beta \leq \pi$) under the Möbius transformation

$$z \mapsto \frac{z+i}{z-i}\,.$$

16 Evaluate the integrals

$$\int_\gamma |z|^2\, dz, \qquad \int_\gamma z^2\, dz$$

when $\gamma : [0,1] \to \mathbb{C}$ is given by (a) $\gamma(t) = e^{i\pi t/2}$, and (b) $\gamma(t) = 1 - t + it$.

17 Show that if $\gamma\colon [\alpha, \beta] \to \mathbb{C}$ is a continuously differentiable curve and $(-\gamma)$ is the opposite curve, then

$$\int_{(-\gamma)} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

18 (Integration by parts) Let f and g are holomorphic in a domain D, and let $\gamma\colon [0,1] \to D$ be a curve with $\gamma(0) = a$, $\gamma(1) = b$. Show that

$$\int_\gamma f(z)g'(z)\, dz = f(b)g(b) - f(a)g(a) - \int_\gamma f'(z)g(z)\, dz.$$

19 Calculate $\int_\gamma z \sin z\, dz$ when γ is the straight line joining 0 to i.

20 Show that the following functions do not have antiderivatives on the domains indicated:

(a) $\dfrac{1}{z}$ $\quad (0 < |z| < \infty)$;

(b) $\dfrac{1}{z} - \dfrac{1}{z-1}$ $\quad (0 < |z| < 1)$;

(c) $\dfrac{z}{1+z^2}$ $\quad (1 < |z| < \infty)$;

(d) $\dfrac{1}{z(1-z^2)}$ $\quad (0 < |z| < 1)$.

Comments and corrections to a.j.scholl@dpmms.cam.ac.uk