COMPLEX ANALYSIS — Example Sheet 2

TKC Lent 2006

The first set of questions are intended to be short and straightforward, the second set are longer, and the final set are additional exercises for those who have completed the earlier ones.

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1. Evaluate $\int_{C(0,2)} \frac{e^z}{z-1}\, dz$ and $\int_{C(0,2)} \frac{e^z}{\pi i - 2z}\, dz$ when $C(0,2)$ is the circle with radius 2, centre 0.
2. Prove Liouville's theorem. The analytic function $f : \mathbb{C} \to \mathbb{C}$ never takes values in the disc $D(c, \varepsilon) = \{z \in \mathbb{C} : |z - c| < \varepsilon\}$. Prove that f is constant.
3. Let D be a domain in $\mathbb{C}$, c a point in D, and $f : D \setminus \{c\} \to \mathbb{C}$ an analytic function. Let $T \subset D$ be a closed triangle with boundary ∂D positively oriented. By dividing T into smaller triangles prove that there is a number R, depending on f but not on T, with
$$\int_{\partial T} f(z)\, dz = \begin{cases} 0 & c \notin T; \\ R & c \in T^\circ. \end{cases}$$
Find the value of R for the functions $f : z \mapsto (z - c)^{-1}$ and $f : z \mapsto (z - c)^{-2}$.
4. Let f be an analytic function on a domain which contains the closed disc $\overline{\mathbb{D}}(z_o, R)$. Prove that
$$|f^{(n)}(z_o)| \leqslant \frac{n!}{R^n} \sup\{|f(z)| : |z - z_o| = R\}\ .$$
Use this to prove Liouville's Theorem.
5. Give an example of an analytic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that has an essential singularity at 0. For this example, find a sequence of points $z_n \to 0$ with $f(z_n) \to 2$ as $n \to \infty$.

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6. Let v and w be two non-zero complex numbers such that v/w is not real. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(z + v) = f(z + w) = f(z)$ for every $z \in \mathbb{C}$. Prove that f is constant.
7. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and satisfy $|f(z)| \leqslant M|z|^\alpha$ for every $z \in \mathbb{C}$ with $|z| > R$, where M, R and α are constants with $0 < \alpha < 1$. Prove that f is constant.
8. Let $f : D \to \mathbb{C}$ be an analytic function on a domain D and let $\overline{B(z_o, R)}$ be a closed disc that lies entirely within D. Let C denote the curve that goes once positively around the boundary of this disc. Prove that
$$f(w) - f(w_o) - \frac{(w - w_o)}{2\pi i} \int_C \frac{f(z)}{(z - w_o)^2}\, dz = \frac{(w - w_o)^2}{2\pi i} \int_C \frac{f(z)}{(z - w)(z - w_o)^2}\, dz\ .$$
for any points $w, w_o \in B(z_o, R)$. Deduce that
$$f'(w_o) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - w_o)^2}\, dz\ .$$
(Does this argument still work if f is only assumed to be continuous and complex differentiable on $D \setminus \{w_o\}$?)
9. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Prove that:
   (a) f is a polynomial of degree at most k if and only if there is a positive constant M such that $|f(z)| \leqslant M(1 + |z|)^k$ for every $z \in \mathbb{C}$.
   (b) f is a non-constant polynomial if and only if $|f(z)| \to \infty$ as $z \to \infty$.
10. (i) Prove that there is no non-constant analytic function defined on $\mathbb{D}$, the open unit disc, that takes only real values.
   (ii) Prove that there is no analytic function f defined on $\mathbb{D}$ with $f(z)^2 = z$ for every $z \in \mathbb{D}$.

11. Let D be a simply connected domain that does not contain 0. Show that there is an analytic function $L : D \to \mathbb{C}$ with $\exp L(z) = z$ for each $z \in D$. (L is an *analytic branch of the logarithm on* D.)

    Show more generally that, for any analytic function $f : D \to \mathbb{C} \setminus \{0\}$, there is an analytic function $\ell : D \to \mathbb{C}$ with $\exp \ell(z) = f(z)$.

12. Let $f : D \to \mathbb{C}$ be an analytic function with the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n$ on a disc $B(z_o, R) \subset D$. Prove that the partial sums of this power series converge uniformly on any compact subset of $B(z_o, R)$. Deduce that

$$\int_0^{2\pi} |f(z_o + re^{i\theta})|^2 \, \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

    for $0 \leqslant r < R$.

    Show that, if $|f(z)|$ achieves its maximum value at z_o, then f is constant.

13. Let f be analytic on the whole of $\mathbb{C}$ except at isolated singularities and suppose that f is one-to-one. Prove the following:
    (i) f has no essential singularity.
    (ii) f has at most one pole.
    (iii) If f has a pole, then the Laurent series at the pole has only a finite number of non-zero coefficients.
    (iv) If the image of f is not $\mathbb{C}$, then f is a Möbius transformation.

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14. Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth function that is never 0. For each $r \geqslant 0$ let $\gamma(r)$ be the curve

$$\gamma(r) : [0, 1] \to \mathbb{C} \; ; \quad t \mapsto f(re^{2\pi it}) \; .$$

    Show that the winding number $n(\gamma(r); 0)$ is 0 for all $r \geqslant 0$.

    Suppose that p is a complex polynomial of degree $N \geqslant 1$. Show that, for r sufficiently large the winding number of the curve $\gamma(r) : [0, 1] \to \mathbb{C}$; $t \mapsto p(re^{2\pi it})$ about 0 is N. Deduce that p has at least one zero.

15. Show that

$$\phi : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C} \setminus [-1, 1] \; ; \quad z \mapsto \frac{z + z^{-1}}{2}$$

    is a conformal map between the two domains. If $f : \mathbb{C} \to \mathbb{C}$ never takes any values in the line segment $[-1, 1]$, show that $\phi^{-1} \circ f$ is analytic and deduce that f must be constant.

16. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function on the entire complex plane. Suppose that, for each point $z \in \mathbb{C}$ there is at least one natural number n with $f^{(n)}(z) = 0$. (The value of n may depend on z. Show that f must be a polynomial.

17. Weierstrass showed that any continuous function on a closed bounded interval $[a, b]$ can be approximated uniformly by polynomials. Can every continuous function on a compact (closed and bounded) subset K of $\mathbb{C}$ be approximated uniformly by polynomials?

18. *(Schwarz' Lemma:)* Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic map with $f(0) = 0$. Show that $g : z \mapsto f(z)/z$ has a removable singularity at 0. Use the maximum modulus principle to deduce that $|f(z)| \leqslant |z|$ for each $z \in \mathbb{D}$. If $|f(z_o)| = |z_o|$ for some $z_o \neq 0$, show that $f(z) = \omega z$ for some ω with $|\omega| = 1$.

19. Let D be a domain containing the line segment $[0, 1]$ and let $f : D \to \mathbb{C}$ be a continuous function that is analytic on $D \setminus [0, 1]$. Prove that f is analytic on all of D.

20. Give an example (with a proof) of an infinitely differentiable function $g : (-1, 1) \to \mathbb{C}$ for which there is no analytic function $f : \mathbb{D} \to \mathbb{C}$ defined on the open unit disc $\mathbb{D}$ with $f(t) = g(t)$ for each $t \in (-1, 1)$.

21. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(z) = z^2$ for each $z \in \mathbb{Z}$. Does it follow that $f(z) = z^2$ for each $z \in \mathbb{C}$?

22. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and suppose that its Taylor series about 0 has infinitely many non-zero coefficients. Let C_R be the curve $C_R : [0, 2\pi] \to \mathbb{C}$; $t \mapsto Re^{it}$. Is it necessarily true that the set of all winding numbers $n(C_R, 0)$, for R with 0 not on C_R, is unbounded?
23. Let ϕ be a closed path. Prove, by differentiating the formula for $n(\phi, z)$, that the winding number is constant on each connected component of $\mathbb{C} \setminus [\phi]$. (You should justify the differentiation.)
24. Let $C : [0, 1] \to \mathbb{C}, t \mapsto \exp 2\pi it$ be the unit circle that divides $\mathbb{C}$ into two parts $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers indexed by $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$.

    (i) Show that the power series
$$s_+(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{(summed only over non-negative integers)}$$
    converges uniformly on the closed disc $\overline{\mathbb{D}}$ to give a function continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$.

    (ii) The function g is defined on the unit circle by $g(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$. Show that g is continuous and that the Cauchy transform:
$$f(w) = \frac{1}{2\pi i} \int_C \frac{g(z)}{z - w}\, dz$$
    exists and is analytic everywhere in $\mathbb{C}$ except on the unit circle.

    (iii) Evaluate $f(w)$ in terms of the coefficients (a_n). Hence show that there are continuous functions
$$s_+ : \overline{\mathbb{D}} \to \mathbb{C}\ ,\ \text{analytic on } \mathbb{D} \qquad \text{and} \qquad s_- : \overline{\mathbb{E}} \to \mathbb{C}\ ,\ \text{analytic on } \mathbb{E}$$
    with $g(z) = s_+(z) - s_-(z)$ for each z on the unit circle.
25. Let Ω be an open connected subset of the Riemann sphere $\mathbb{C}_\infty$ that contains ∞. We say that a function $f : \Omega \to \mathbb{C}_\infty$ is *complex differentiable at* ∞ if the map $z \mapsto f(1/z)$ is complex differentiable at 0. We say that f has a *pole at* ∞ if $z \mapsto f(1/z)$ has a pole at 0. Finally, f is *meromorphic* if it is complex differentiable or has a pole at each point of Ω. Prove that f is complex differentiable at ∞ if and only if $f(z)$ tends to a finite limit as $z \to \infty$. Also, f has a pole at ∞ if and only if $f(z) \to \infty$ as $z \to \infty$. Show that each rational function is an analytic function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$.

    Suppose that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a meromorphic function on the entire Riemann sphere. Show that each pole of f is isolated and deduce that f has only a finite number of poles. Deduce that f is a rational function.
26. Let $f : D \to \mathbb{C}$ be an analytic function on a domain D that contains the closed unit disc $\overline{\mathbb{D}}$. Let C be the unit circle. For $w \in \mathbb{D}$ with inverse point $w^* = 1/\overline{w}$ explain why
$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w}\, dz = \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta} - w} \frac{d\theta}{2\pi} \qquad \text{and}$$
$$0 = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w^*}\, dz = \int_0^{2\pi} f(e^{i\theta}) \frac{-\overline{w}}{e^{-i\theta} - \overline{w}} \frac{d\theta}{2\pi}\ .$$
    Deduce *Poisson's formula*:
$$f(w) = \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi}\ .$$
    Show that the Poisson kernel $P_w(e^{i\theta}) = \dfrac{1 - |w|^2}{|e^{i\theta} - w|^2}$ is a probability distribution on C for each w in the unit disc. (That is, $P_w(e^{i\theta}) \geqslant 0$ and $\displaystyle\int_0^{2\pi} P_w(e^{i\theta})\, \frac{d\theta}{2\pi} = 1$.)
27. Let D be a domain in $\mathbb{C}$ with a disconnected complement. Show that there are two disjoint open sets C, C' with $C \cup C' = \mathbb{C} \setminus D$ and C compact. Show that there is a $\delta > 0$ with $|z - w| > \delta$ for each $z \in C$ and $w \in C'$.

    Choose a point $c \in C$ and cover $\mathbb{C}$ a grid of squares of side length $\frac{1}{4}\delta$ with c in the centre of one of the squares. Let K be the union of all the closed $\frac{1}{4}\delta \times \frac{1}{4}\delta$ squares in this grid that meet C. Show that the boundary ∂K consists of sides of the squares that lie entirely within D. Deduce that there is a piecewise continuously differentiable closed curve γ in D with $n(\gamma; c) \neq 0$.

+28. Let D be a domain in $\mathbb{C}$. Show that the following are equivalent:

(a) D is simply connected.

(b) Every piecewise smooth closed curve in D has winding number 0 about any point $c \in \mathbb{C} \setminus D$.

(c) $\mathbb{C} \setminus D$ has no bounded components.

[*Hint: It suffices to consider closed curves made up of straight line segments parallel to the axes.*]

29. Let $f : D \to \mathbb{C}$ be an analytic function with $f'(z_o) \neq 0$. Let C be a circle of suitably small radius. Show that

$$h(w) = \frac{1}{2\pi i} \int_C \frac{zf'(z)}{f(z) - w} \, dz$$

defines an analytic function on some neighbourhood of $f(z_o)$ that is inverse to f.

Give an example to show that f need not have an inverse on all of $f(D)$ even if f' is never 0 on D.

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