Complex analysis exercises 4

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1. Let $\mathcal{A} \subset \mathbb{C}$ be an open set, $F : \mathcal{A} \times [0,1] \longrightarrow \mathbb{C}$ such that F is continuous (on $\mathcal{A} \times [0,1]$) and $z \mapsto F(z,s)$ is holomorphic for every $s \in [0,1]$. Then

$$f : z \mapsto \int_0^1 F(z,s)ds$$

is holomorphic on $\mathcal{A}$.

2. (a) Calculate explicitly (by parametrization of the circle) the integral

$$\int_{S^1} \frac{1}{z - z_0} dz$$

where z_0 is in the disc centered at the origin and radius 1 and $S^1 = \{z : |z| = 1\}$ is the boundary circle. Do the same calculation using the keyhole contour.

(b) Calculate $\int_{S^1} \sqrt{z} dz$ where $\sqrt{z}$ is the principal branch of the square root (i.e. with branch cut on the negative real axis) by (i) explicit parametrization, and (ii) use of a keyhole contour and Cauchy's theorem to relate it to a real integral from -1 to 0 on the real axis.

3. Show that if f is an analytic function then its set of zeros $\{z : f(z) = 0\}$ necessarily consists of isolated points.

Consider a meromorphic function on $\mathbb{C}$ with the property that $\lim_{|z|\to\infty} |f(z)| = +\infty$. Show that it cannot have poles at all integer points $z = n$ on the real axis.

4. Suppose f, g are analytic in a region containing $\{z : |z| \leq 1\}$ and f has a simple zero at $z = 0$ and vanishes nowhere else in $\{z : |z| \leq 1\}$. Let $f_\epsilon(z) = f(z) + \epsilon g(z)$ for real ϵ. Show that for sufficiently small ϵ (i) f_ϵ has a unique zero z_ϵ in $\{z : |z| \leq 1\}$, and (ii) z_ϵ is a continuous function of ϵ.

5. Prove that all entire (i.e. analytic on the whole of $\mathbb{C}$) functions that are also injective take the form $f(z) = az + b$ for some complex numbers a, b. (Hint: apply Casorati-Weierstrass to $g(z) = f(1/z)$).

6. State the residue theorem. Evaluate

$$lim_{R\to\infty} \int_{-R}^{R} \frac{x \sin \pi x}{x^2 + 2x + 5} dx.$$

7. If f is analytic in the punctured disc $0 < |z| < 1$ and satisfies $|f(z)| \leq C|z|^{-1+\epsilon}$ for some positive number ϵ then 0 is a removable singularity for f.

8. Let D be a domain in $\mathbb{C}$ and $\gamma : [0, 1] \to D$ an arbitrary *continuous* path.

(i) Show that there is an $r > 0$ such that $B(\gamma(t), r) \subseteq D$ for all $t \in [0, 1]$.

(ii) Show that we may construct a piecewise continuously differentiable path $\delta : [0, 1] \to D$ such that $\delta(0) = \gamma(0)$, $\delta(1) = \gamma(1)$ and $|\delta(t) - \gamma(t)| < r$ for all $t \in [0, 1]$. [Hint: subdivide $[0, 1]$ into intervals of length $1/n$, where $1/n$ is a Lebesgue number for the covering $\{\gamma^{-1}(B(\gamma(t), r/2)) \mid 0 \leq t \leq 1\}$.]

(iii) Show that any two paths δ_1, δ_2 satisfying the conditions of (ii) are homotopic in D, by a homotopy which fixes the end-points $\gamma(0)$ and $\gamma(1)$. Deduce that $\int_{\delta_1} f(z)\, dz = \int_{\delta_2} f(z)\, dz$ for any analytic function $f : D \to \mathbb{C}$. [This allows us to extend the definition of $\int_\gamma f(z)\, dz$ to the case when γ is (continuous but) not piecewise continuously differentiable.]

9. Let D be a bounded domain such that $\mathbb{C} \setminus D$ is disconnected.

(i) Show that just one component of $\mathbb{C} \setminus D$ is unbounded.

(ii) Now assume that $\mathbb{C} \setminus D$ has finitely many components; let C be a bounded component, and let $C' = (\mathbb{C} \setminus D) \setminus C$. Show that there is a $\delta > 0$ such that $|z - w| \geq \delta$ for all $z \in C$ and $w \in C'$.

(iii) Now let $c \in C$ and cover $\mathbb{C}$ by a grid of squares of side $\delta/2$ with c in the centre of one of the squares. Let K be the union of all the (closed) squares in this grid which meet C. Show that the boundary ∂K lies entirely in D. Deduce that there is a simple closed contour γ lying in D with $n(\gamma, c) = 1$.

10. (i) Let $f_n(z) = \sum_{m=-n}^{+n} (z - m)^{-2}$. Show that the sequence $(f_n(z))$ converges pointwise on $\mathbb{C} \setminus \mathbb{Z}$ to a function $f(z)$. Show also that $f(z + 1) = f(z)$ for all z.

(ii) Show that the convergence in (i) is uniform on any set of the form $\{z \mid |z| < R, z \notin \mathbb{Z}\}$. Deduce that $f(z)$ is an analytic function on $\mathbb{C} \setminus \mathbb{Z}$.

(iii) Now let $g(z) = \pi^2 \operatorname{cosec}^2 \pi z$. Show that the function $f(z) - g(z)$ has only removable singularities in $\mathbb{C}$. Show also that $f(x+iy)$ and $g(x+iy)$ both tend to 0 as $y \to \pm\infty$ [hint: $|\sin^2(x + iy)| = \sin^2 x + \sinh^2 y$], and deduce that $f(z) = g(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

(iv) By considering the constant terms in the Laurent expansions of f and g about $z = 0$, deduce that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

11. (Tripos 1999 IV 13. Hadamard 3 circles theorem.) Let f be analytic in a domain containing the annulus $\{z \mid r_1 \leq |z| \leq r_2\}$. By applying the maximum modulus principle to functions of the form $z^p(f(z))^q$ for suitable p and q, prove that

$$M(r)^{\log(r_2/r_1)} \leq M(r_1)^{\log(r_2/r)}\, M(r_2)^{\log(r/r_1)}$$

for all $r \in (r_1, r_2)$, where $M(r)$ denotes $\sup\{|f(z)| \mid z \in C(0, r)\}$. Deduce that $\log M(r)$ is a convex function of $\log r$. [Recall that a real function is said to be *convex* if the chord joining any two points on its graph lies above, or coincides with, the graph.]