Complex analysis exercises 3

Sophia Demoulini Lent 2005

1. (Inverse function theorem for holomorphic functions.) Let f be an analytic function on an open set D, and suppose $f'(z_0) \neq 0$. Show that for sufficiently small r the formula

$$g(w) = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{z\, f'(z)}{f(z) - w}\, dz$$

defines an analytic function on some neighbourhood of $f(z_0)$ which is inverse to f. Give an example to show that f need not have an inverse on the whole of its image even if f' is never zero on D.

2. Sketch the closed contour $\gamma\colon [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = 4e^{3it} \cos 2t$, and verify that $n(\gamma, 3) = 1$. What is $n(\gamma, 1)$?

3. (i) Let f be entire and never taking the value 0. For each $r \geq 0$, let $\gamma(r, f)$ be the closed contour defined by $\gamma(r, f)(t) = f(re^{2\pi it})$. Show that $n(\gamma(r, f), 0) = 0$ for all $r \geq 0$.
(ii) Let $p(z)$ be a complex polynomial of degree $N \geq 1$. Show that for sufficiently large values of r we have $n(\gamma(r, p), 0) = N$. Deduce that $p(z)$ has at least one zero.

4. (Tripos 1992 I 4, adapted) (i) Let $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$ be a polynomial in z of degree $k > 0$. Show that there are positive real constants M, N and R such that $N|z|^k \leq |p(z)| \leq M|z|^k$ whenever $|z| \geq R$.
(ii) Let $g(z) = p(z)/q(z)$ be a rational function, where $\deg q \geq \deg p + 2$. Show that the sum of the residues of g at all its singularities is zero.
(iii) Hence or otherwise evaluate $\displaystyle\int_\gamma \frac{(z-1)^2(z+2i)^2}{z^4(z-2)(z+2)}\, dz$ where γ is the contour $t \mapsto e^{2\pi it}$.

5. Let f be an analytic function having a zero of order n at $z = 0$ (i.e., the first non-vanishing term in its Taylor series is $a_n z^n$). Prove that there are open neighbourhoods U and V of 0 such that f is an n-to-1 mapping from $U \setminus \{0\}$ to $V \setminus \{0\}$ (i.e., for each $w \in V \setminus \{0\}$, the equation $f(z) = w$ has n distinct solutions in $U \setminus \{0\}$).

6. In each of the following cases, find the number of zeros of the given function f in the given domain D:
(i) $f(z) = z^7 - 2z^6 + 6z^3 - z + 1$, $D = B(0, 1)$;
(ii) $f(z) = z^4 - 6z + 3$, $D = \{z \mid 1 < |z| < 2\}$;

7. (Tripos 2003 IV 13) Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for each such z. Hence sketch the contour $p \circ \gamma$, where $\gamma(t) = e^{2\pi it}$, and use your sketch to determine, for each real value of t, the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$.

8. Let $\phi \in C^\infty(\mathbb{R})$ be absolutely integrable with $\int \phi(x)dx = 1$. Define $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$ and show

$$\phi_\epsilon * f(x) - f(x) = \int \big(f(x - \epsilon w) - f(x)\big)\phi(w)dw$$

(where $*$ is convolution and the integrals are over $\mathbb{R}$). Now deduce that if $f : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous then $\phi_\epsilon * f(x) \to f(x)$ as $\epsilon \to 0$ uniformly. [Hint: split up the w integral into an integral over the ball $B_R = \{|w| < R\}$ and its complement B_R^c for large R and use the fact that $\displaystyle\lim_{R\to+\infty} \int_{|x|>R} |\phi(x)|dx = 0$.]

In the following questions the Fourier transform of a Schwartz function f is the function $\hat{f}(\xi) = \int e^{-2\pi ix\xi} f(x)dx$, the integral being over $\mathbb{R}$.

9. Compute the Fourier transform $\hat{G}_\epsilon$ of the function $G_\epsilon(x) = e^{-\epsilon^2 x^2}$ for $\epsilon > 0$. Show that

$\hat{G}_\epsilon(\xi) = \epsilon^{-1}\hat{G}_1(\xi/\epsilon)$. Compute the Fourier transform of $F_{x_0}(x) = f(x - x_0)$ for any x_0, f Schwartz.

10. (Plancherel-Parseval) Prove that if F, G are Schwartz functions with Fourier transforms $\hat{F}, \hat{G}$ then

$$(*) \qquad \int \hat{F}(y)G(y)dy = \int F(x)\hat{G}(x)dx$$

(where the integrals are over $\mathbb{R}$). *You may assume without proof that interchange of the order of integration in the repeated integrals that arise after substitution of the definitions of $\hat{F}, \hat{G}$ into $(*)$ is allowed (Fubini).* Now apply (*) in the case $G = G_\epsilon(x) = e^{-\epsilon^2 x^2}$ and $F(x) = f(x - x_0)$ and deduce by taking the limit $\epsilon \to 0$ (and using the previous two questions) that if f is Schwartz then $f(x) = \int e^{2\pi i x y}\hat{f}(y)dy$ (Fourier inversion).

11. A doubly periodic funtion which is not constant cannot be holomorphic on $\mathbb{C}$ (exercise 11, sheet 1). A non-constant doubly periodic meromorphic function is called *elliptic*. Let $v, w \in \mathbb{C}$ linearly independent and let P be the period parallelogram $\{z : z = \lambda v + \mu w,\ 0 \leq \lambda, \mu < 1\}$. If f is elliptic then show that
(i) the total number of poles in P is at least 2.
(ii) f has the same number of zeros as poles in P.
(iii) For any $c \in \mathbb{C}$ the equation $f(z) = c$ has as many solutions as f has poles in P. [If a zero or pole lies on ∂P, then translate P.]

12. The Cauchy integral formula (and the formula about derivatives) and the residue theorem were proved under the assumption that the integrals involved were taken over a circle, parametrised to be positively traversed once. How do these formulas change if these integrals are instead taken to be any piecewise C^1 closed curve within a star domain?

13. If f is analytic on an open annulus $\mathcal{A}$ then the integral of f is the same on any circle in $\mathcal{A}$ concentric with the annulus (when the circle is positively parametrised and traversed once). [This follows from the general Cauchy theorem; here it suffices to show e.g. by construction of a primitive, that the Cauchy theorem (proved for star domains) also holds for a keyhole contour which is the boundary of $\{\rho < |z| < r\} \setminus \{z = x + iy : -\epsilon < x < \epsilon, 0 < y < r\}$].

14. Evaluate the integrals (suggestions of some contours were given in class)
(i) $\int_0^\infty sinx^2\ dx$
(ii) $\int_0^\infty \frac{x^{a-1}}{1+x}\ dx$, for $0 < a < 1$
(iii) [Fourier Transform of the Gaussian] $\int_{-\infty}^\infty e^{-ax^2}e^{-i\xi x}dx$ with a positive. [Complete the square in the exponent; use a rectangular contour with one side on the real axis]
(iv) $\int_0^\infty \frac{ln(x^2+1)}{x^2+1}dx$ [use an upper semicircle]
(v) $\int_0^\infty \frac{cosh\ ax}{cosh\ x}dx$, $|a| < 1$ [use a rectangular contour with one side on the real axis.]

15*. [Tripos, adapted]. Define the Hermite polynomials $H_n(x)$ by $e^{2xz - z^2} = \sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}$ and the Hermite functions by $h_n(x) = H_n(x)e^{-\frac{x^2}{2}}$. Prove that
(i) $H_n(x) = (-1)^n e^{x^2}(\frac{d}{dx})^n e^{-x^2}$
(ii) $\int_{-\infty}^{+\infty} f(x)h_n(x)dx = 0 \forall n$ implies $f \equiv 0$
(iii) $\int_{-\infty}^{+\infty} h_m(x)h_n(x)dx = \delta_{nm}\sqrt{\pi}2^n n!$
(iv)the functions $\eta_n(x) = h_n(\sqrt{2\pi}x)$ are eigenfunctions of the Fourier transform : $\hat{\eta}_n(\xi) = (-i)^n\eta_n(\xi)$

Can you develop a proof of the Fourier inversion theorem on the Schwartz space based on these properties.