Complex analysis exercises 2

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1. Consider the function $\phi(z) = \frac{1}{2}(z + z^{-1})$ and find the points of non-conformality in $\mathbb{C}$; at these points describe the action of ϕ on angles. [If $J(z) = \dfrac{z-1}{z+1}$ then $J(\phi(z)) = (J(z))^2$.] Show that ϕ defines a (bijective) conformal mapping of $\{z \in \mathbb{C} \mid |z| > 1\}$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$. (This map can be used as the map ϕ in exercise 10 of sheet 1.)

2. Evaluate $\displaystyle\int_{C(0,2)} \frac{e^z}{z-1}\,dz$ and $\displaystyle\int_{C(0,2)} \frac{e^z}{\pi i - 2z}\,dz$ when $C(0,2) = \{|z| = 2\}$.

3. Prove that if f is holomorphic in an open set which contains the closure of $B = \{z : |z - z_0| < r\}$ then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} max_{z\in\partial B}|f(z)|.$$

Use this to give another proof of Liouville's theorem.

4. (i) Give an example of an infinitely differentiable function $f\colon (-1,1) \to \mathbb{R}$ for which there exists a domain D containing $(-1,1)$ and an analytic function $g\colon D \to \mathbb{C}$ which extends f (i.e. $g(t) = f(t)$ for all $t \in (-1,1)$), but we cannot take D to be the disc $B(0,1)$.
(ii) Give an example of an infinitely differentiable function $f\colon (-1,1) \to \mathbb{R}$ which cannot be extended to an analytic function on any open set in $\mathbb{C}$ containing $(-1,1)$.
(iii) Is the $\mathbb{R}^2$ analogue of the theorem on uniform convergence of complex differentiable functions valid, i.e. is the uniform limit of real (continuously) differentiable functions on $\mathbb{R}^2$ (continuously) differentiable?

5. (i) If $f(z)$ is analytic in $\mathbb{C}$ and there exist constants M and R such that $|f(z)| \leq M|z|^k$ whenever $|z| \geq R$, then f is a polynomial of degree at most k if k is a positive integer and a constant if $0 < k < 1$. [Consider Cauchy's integral formula for the Taylor coefficients of f.]
(ii) If $f(z)$ is analytic in $\mathbb{C}$ and there exist constants $N > 0$ and R such that $|f(z)| \geq N|z|^k$ whenever $|z| \geq R$, then f is a polynomial of degree at least k. [Consider the singularity of $f(1/z)$ at $z = 0$.]

6. Let f be an entire function and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion $f(z) = \displaystyle\sum_n^{\infty} a_n(z - z_0)^n$ is zero. Prove that f is a polynomial.

7. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(z) = z^2$ for all $z \in \mathbb{Z}$. Does it follow that $f(z) = z^2$ for all $z \in \mathbb{C}$?

8. Every continuous function on a compact interval $[a,b]$ can be uniformly approximated by polynomials (Weierstrass' theorem). Can every continuous fucntion on a compact (closed and bounded) $K \subset \mathbb{C}$ be approximated uniformly by polynomials?

9. Let f be an analytic function on an open connected set containing the closed disc $\overline{B(0,1)}$. If f maps $\overline{B(0,1)}$ into itself with $f(0) = 0$, then the maximum modulus principle to show that $|f(z)| \leq |z|$ for all $z \in \overline{B(0,1)}$.

10. Is there a function holomorphic on the open unit disc and continuous on the closed unit disc such that $f(z) = \dfrac{1}{z}$ on the unit circle? [The circle is not known to be within the region of analyticity.]

11. (i) Show that a differentiable function in an open disc D has a primitive on D (without using Cauchy's theorem for star domains). [This was done in the proof of Morera's theorem].
(ii) Show that the Cauchy theorem holds for slightly more general than star domains: let f be holomorphic in an open set in $\mathbb{C}$ containing the set $K = \{z = x + iy : |x| \leq a, |y| \leq a\} \setminus \{z =$

$x + iy : |x| < \epsilon, \epsilon < y \leq a\}$ for positive contants $a, \epsilon < a$. Then $\displaystyle\int_{\partial K} f(z)dz = 0$. [Construct a primitive]. (Similarly for any closed smooth curve γ in K with "interior" inside K.)

12. Let $B = B(0,1)$ denote the open unit disc, $C = C(0,1)$ the unit circle and D the set $\{z \in \mathbb{C} \mid |z| > 1\}$. Suppose given a doubly infinite sequence $(a_n \mid n \in \mathbb{Z})$ of complex numbers such that $\sum_{n=-\infty}^{+\infty} |a_n|$ converges.

(i) Show that the sum

$$\sum_{n=-\infty}^{+\infty} a_n z^n = \lim_{M,N\to\infty} \sum_{n=-M}^{+N} a_n z^n$$

converges uniformly on C to a continuous function $g(z)$.

(ii) Consider the Cauchy transform $f(w) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{g(z)}{z-w}\,dz$ where γ is the contour $t \mapsto e^{2\pi i t}$. Show that f is analytic on $\mathbb{C} \setminus C$.

(iii) Evaluate $f(w)$ in terms of the coefficients a_n, distinguishing between the cases $|w| < 1$ and $|w| > 1$. Hence show that we can write $g(z) = g_+(z) + g_-(z)$, where g_+ is a continuous function on the closed disc $\overline{B}$, analytic on B, and g_- is continuous on $\overline{D}$ and analytic on D.

13. If f is a function analytic in $\{z \mid |z| > R\}$ for some R, we define the *singularity of f at* ∞ to be the singularity of the function $z \mapsto f(1/z)$ at 0. In particular, we say f is *analytic at* ∞ if $z \mapsto f(1/z)$ has a removable singularity at 0. A function is said to be *meromorphic* if its only singularities in the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ are (removable or) poles.

(i) Show that f is analytic at ∞ if and only if $f(z)$ tends to a finite limit as $z \to \infty$, and f has a pole at ∞ if and only if $f(z) \to \infty$ as $z \to \infty$.

(ii) Prove that a rational function (that is, a function of the form $f(z) = p(z)/q(z)$, where p and q are polynomials and q is not identically 0) is meromorphic.

(iii) Show that a meromorphic function f has only finitely many poles. Deduce that there is a polynomial $q(z)$ such that $z \mapsto q(z)f(z)$ has only removable singularities in $\mathbb{C}$, and hence show that f is a rational function.

14. Let f be analytic on the whole of $\mathbb{C}$ except for isolated singularities and suppose that f is one-to-one. Prove the following:

(i) f has no essential singularities;

(ii) f has at most one pole in $\mathbb{C}$, and if it has one then it is analytic at ∞ (the Laurent series at the pole has only a finite number of non-zero coefficients).

(iii) f is a Möbius transformation.

15. (Fourier coefficients) Let f be analytic on a disc $B(z_0, r) = \{z : |z - z_0| < r\}$. Then the coefficients of the power series expansion of f for $0 < \rho < r$ are given by

$$a_n = \frac{1}{2\pi\rho^n}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})e^{-in\theta}d\theta$$

if n is a non-negative integer. Show also that the integral above is zero if $n \in \mathbb{Z}^-$.

16. Let f be a holomorphic function on the disc centre 0 radius R. (i) Show that if $0 < r < R$ and $|z| < r$ then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})Re\Big(\frac{re^{i\theta}+z}{re^{i\theta}-z}\Big)\,d\theta$$

(ii) Show that

$$Re\frac{re^{i\theta}+\rho}{re^{i\theta}-\rho} = \frac{r^2-\rho^2}{r^2 - 2r\rho cos\theta + \rho^2}$$

[If $w = \frac{r^2}{\bar{z}}$ then $\displaystyle\int \frac{f(z)}{z-w}\,dz$ around the boundary of the disc is zero. Use this with the Cauchy integral formula].