Complex analysis exercises 1

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1. Show that complex integration is a linear operation over $\mathbb{C}$ on the space of continuous functions f on a piecewise continuously differentiable path γ in $\mathbb{C}$. Show the properties
(i) $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$
(ii) $\int_{-\gamma} f(z)\, |dz| = \int_\gamma f(z)\, |dz|$
(iii) $|\int_\gamma f(z)\, dz| \le \int_\gamma |f||dz|$

2. Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ that is infinitely differentiable, when considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, but its integral around the unit circle is 1.

3. Which of the following subsets of $\mathbb{C}$ are simply connected? Justify your answers.
(i) $\mathbb{C} \setminus (\{x \in \mathbb{R} \mid |x| \ge 1\} \cup \{iy \mid y \in \mathbb{R}, |y| \ge 1\})$.
(ii) $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.
(iii) $\{z \in \mathbb{C} \mid |z| > 1, |z+1| < 2\}$.

4. Show that the maps $z \mapsto \sqrt{\frac{i(1-z)}{z+1}}$ and $z \mapsto \frac{z-a}{1-z\bar{a}}$ for $a \in \mathbb{C}$ are conformal and find the image of the unit disk under each.

5. Given a domain (i.e. open connected set) $D \subseteq \mathbb{C}$, verify that the function $d(\gamma, \delta) = \sup\{|\gamma(t) - \delta(t)| \mid t \in [0,1]\}$ defines a metric on the set $K(D)$ of closed contours (i.e. piecewise cont/ly diff/able paths) defined on $[0,1]$ in D. Show that D is simply connected if and only if $K(D)$ is path-connected, and that $K(D)$ is path-connected if and only if it is connected.

6. Consider the domain $\mathcal{A} = \mathbb{C} \setminus \mathcal{S}$ where $\mathcal{S}$ is a continuous path which starts at the origin and is the graph of the equation $r = \theta$ (in polar co-ordinates, $arg z = \theta \in [0, \infty)$ increasing). Use the harmonic function $u(x,y) = log(\sqrt{x^2+y^2})$ to show there is a choice of $arg z$ (obtained as a conjugate harmonic to u) so that the logarithm is analytic on $\mathcal{A}$. Do the same for the logarithmic branch $\mathbb{C} \setminus \{z : Re\ z \le 0\}$ and compare. Is $\mathrm{Im}(log z)$ bounded on $\mathcal{A}$?

7. Let $D \subseteq \mathbb{C}$ be a simply connected domain which does not contain 0. Show that there is an analytic function $L\colon D \to \mathbb{C}$ (an *analytic branch* of the logarithm function) such that $e^{L(z)} = z$ for each $z \in D$. [Hint: define $L(z)$ as the integral of $1/z$ along a suitable contour.] More generally, if $f\colon D \to \mathbb{C} \setminus \{0\}$ is any analytic function, show that there is an analytic function $g\colon D \to \mathbb{C}$ with $e^{g(z)} = f(z)$.

8. (i) Show that the map $t \mapsto e^{it}$ is a continuous bijection on any interval $[a, a+2\pi)$ onto S^1, the unit circle in $\mathbb{C}$ and a homeomorphism (i.e. a continuous bijection with continuous inverse) of $(a, a+2\pi)$ on the complement of e^{ia} in S^1.

(ii) A continuous funtion on a metric space X into S^1 is called *inessential* if there is a continuous real function g such that $f(x) = e^{ig(x)}$ for every $x \in X$. A continuous $f : X \to S^1$ is called *essential* if it is not inessential. Show that any continuous $f : X \to S^1$ such that $f(X) \ne S^1$ is inessential.

9. Let D be an open subset of $\mathbb{R}^2$. A function $u\colon D \to \mathbb{R}$ is said to be *harmonic* on D if it is twice continuously differentiable and satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere on D. Show that the real part of any analytic function is harmonic. You will need to assume the fact that a complex differentiable function is infinitely differentiable (to follow later from the Cauchy theorem). Conversely, suppose u is harmonic on D; show that the function

$$\phi(x+iy) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y)$$

is analytic on D, and deduce that if D is simply connected then u is the real part of an analytic function defined on D. Give an example to show that the hypothesis 'simply connected' cannot be weakened to 'connected'.

10. (i) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant analytic function. Show that the image $\{f(z) \mid z \in \mathbb{C}\}$ of f is dense in $\mathbb{C}$ (i.e., has nonempty intersection with every open disc $B(z,r)$, $r > 0$).

(ii) Let $\mathcal{A}$ be a domain containing the closed unit disk and suppose $\phi : (\bar{\mathcal{A}})^c \longrightarrow \mathbb{C} \setminus [a,b]$ is an analytic bijection with analytic inverse (where $[a,b]$ is a real interval). Then if f is analytic and $f(\mathbb{C}) \subset \mathbb{C} \setminus [a,b]$ then f is constant.

11. (i) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function and suppose there are nonzero complex numbers v, w such that $f(z+v) = f(z) = f(z+w)$ for all $z \in \mathbb{C}$ and v/w is non-real. Prove that f is constant.
(ii) How about the case of $\frac{v}{w} \in \mathbb{R}$? (Consider $\frac{v}{w}$ in $\mathbb{Q}$ and in $\mathbb{R} \setminus \mathbb{Q}$ separately).

12. Let $f\colon D \to \mathbb{C}$ be an analytic function on a domain D, and let $z_0 \in D$. Let γ be the (positively oriented) boundary of a closed disc $\overline{B(z_0,r)}$ lying within D. Prove that

$$f(z) - f(z_0) - \frac{(z-z_0)}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^2}\,dw = \frac{(z-z_0)^2}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)(w-z_0)^2}\,dw$$

for any $z \in B(z_0,r)$. Deduce that

$$f'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^2}\,dw\ .$$

13. Show that any **real** linear map $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$ can be written as $T : z \mapsto Az + B\overline{z}$ for two complex numbers A and B. Then T is complex linear if and only if $B = 0$.

Suppose that $f : D \to \mathbb{C}$ is a **real** differentiable function at the point $z_0 \in D$. Show that we can write the derivative $f'(z_o)$ as

$$f'(z_o) : z \to Az + B\overline{z}\ .$$

We will write $\dfrac{\partial f}{\partial z}(z_o)$ for A and $\dfrac{\partial f}{\partial \overline{z}}(z_o)$ for B. (In spite of the notation, these are NOT partial derivatives.) Find a formula for $\dfrac{\partial f}{\partial z}(z_o)$ and $\dfrac{\partial f}{\partial \overline{z}}(z_o)$ in terms of the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at z_o. Show that f is analytic if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$ at each point of D.

14. (Cauchy's theorem under the assumption of continuous differentiability) Let $f : D \to \mathbb{C}$ be a function on a domain D that has continuous partial derivatives of any order but may not be complex differentiable. Suppose that the rectangle $R = \{x + iy \| a_1 \le x \le a_2\ ,\ b_1 \le y \le b_2\}$ lies within D and that ∂R is the boundary curve of R, positively oriented.

(i) Prove that

$$\int_{a_1}^{a_2} \frac{\partial f}{\partial x}(x+iy)\,dx = f(a_2+iy) - f(a_1+iy)$$

for $b_1 \le y \le b_2$.

(ii) Deduce that

$$\int_R \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dx\,dy = -i\int_{\partial R} f(z)\,dz$$

and hence that

$$\int_R \frac{\partial f}{\partial \overline{z}}\,dx\,dy = \frac{-i}{2}\int_{\partial R} f(z)\,dz\ .$$

(iii) Show that $\int_{\partial R} f(z)\,dz = 0$ for all rectangles R within D if and only if f is complex differentiable at each point of D.