ANALYSIS AND TOPOLOGY—EXAMPLES 3

(updated 15 November 2024)

Exercises

1. Recall from lectures that a topological space (X, τ) is said to be Hausdorff if $\forall x, y \in X, x \neq y \implies \exists U, V \subset X$ s.t. $x \in U, y \in V, U \cap V = \emptyset$.

(a) Show that if X is finite there is only one topology (up to equivalence of topologies) on X which is Hausdorff.

(b) Show that if (X, τ) is metrizable then it is Hausdorff.

(c) Show that if (X, τ) is Hausdorff and (x_n) is a convergent sequence on X then its limit is unique.

2. Let $\mathrm{id}\colon (X, \tau) \to (X, \rho)$ be the identity function on a set X endowed with possibly different topologies τ and ρ. Is id necessarily continuous?

3. Show that the Hausdorff property and second countability are topological invariants.

4. Show that the metric topology on $\mathbb{R}^n$ induced by the Euclidean metric is equivalent to the product topology inherited by the Euclidean metric on $\mathbb{R}$.

Problems

5. Let (X, τ) and (Y, ρ) be topological spaces with (Y, ρ) Hausdorff.

(a) Show that $\Delta = \{(y, y)\colon y \in Y\}$ is a closed subset of $Y \times Y$ with the product topology.

(b) Let $f, g\colon X \to Y$ be continuous. Show that $\{x \in X\colon f(x) = g(x)\}$ is a closed subset of X.

6. ($\star$) The goal of this problem is to investigate, for a topological space, under what conditions the closure is the same as the sequentially closure of a set.

(a) Show that, if X is uncountable and $\tau = \emptyset \cup \{A \subset X\colon X \backslash A \text{ countable}\}$ is the co-countable topology, some sequentially closed subsets of (X, τ) are not closed.

(b) Show that, if (X, τ) is second countable, a subset $A \subset X$ is sequentially closed iff it is closed.

7. ($\star$) Let (X, τ) and (Y, ρ) be topological spaces. Recall from lectures that a function $f\colon X \to Y$ is said to be sequentially continuous if for every convergent sequence (x_n) in X with $x_n \to x$, we have $f(x_n) \to f(x)$. The goal of this question is to investigate the relationship between continuity and sequential continuity.

(a) Show that continuous functions $f\colon X \to Y$ are sequentially continuous.

(b) Give an example of (X, τ) and (Y, ρ) and $f\colon X \to Y$ such that f is sequentially continuous but not continuous.

(c) Show that, if (X, τ) is second countable, then sequentially continuous functions $f\colon X \to Y$ are continuous.

8. Let (X, τ) be a topological space, $X = A \cup B$ for $A, B \subset X$. Let (Y, ρ) be another topological space. Take $g\colon A \to Y$ and $h\colon B \to Y$ be continuous functions, with respect to the subspace topology, which agree on $A \cap B$. Define a function $f\colon X \to Y$ by

$$f(x) = \begin{cases} g(x), & x \in A \\ h(x), & x \in B \end{cases},$$

which "glues" together g and h.

(a) Show that, if A and B are both closed in X, then f is continuous.

(b) Is this still true if A is not closed in X?

9. Show that a finite product of metrizable topological spaces is metrizable. *Hint: start by considering the product of two metric spaces* (X, d) *and* (Y, e)*, and defining an appropriate metric on* $X \times Y$*.*

10. Let $X = [0, 1] \cup [2, 3]$ with the usual topology and define an equivalence relation $\sim$ on X by $x \sim y$ iff $x = y$ or $\{x, y\} = \{1, 2\}$. Show that $X/\sim$ is homeomorphic to $[0, 2]$ with the usual topology.

11. Let X, Y be topological spaces and endow $X \times Y$ with the product topology.

(a) Show for each $y \in Y$ that $X \times \{y\}$, as a subspace of $X \times Y$, is homeomorphic to X.

(b) Define an equivalence relation $\sim$ on $X \times Y$ by $(x, y) \sim (x', y')$ iff $x = x'$. Assuming Y is non-empty, show that $X \times Y/\sim$ is homeomorphic to X.

12. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (z, w)$ iff $x - z \in \mathbb{Z}$ and $y - w \in \mathbb{Z}$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to the torus.

$$T = \left\{ ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta) \middle| \theta, \phi \in [0, 2\pi] \right\}.$$

13. Let $X = \{(x, y) \in \mathbb{R}^2 \colon y = \pm 1\}$ and $\sim$ be the equivalence relation on X defined by $(x, y) \sim (x', y') \Leftrightarrow x' = x \neq 0, y' = -y$. Define $Q = X/\sim$.

(a) Show that every point in Q has a neighborhood homeomorphic to $\mathbb{R}$. *Hint: it helps to sketch* Q*.*

(b) Show that Q is, however, not Hausdorff.

14. Recall that, on a set X, we say that $\tau = \emptyset \cup \{A \subset X \colon X \backslash A \text{ countable}\}$ is the cocountable topology and $\rho = \emptyset \cup \{A \subset X \colon X \backslash A \text{ finite}\}$ is the cofinite topology. Can (X, τ) be compact? What about (X, ρ)?

15. ($\star$) Let X be a compact Hausdorff space. Show that disjoint closed subsets of X can be separated: if F_1, $F_2 \subset X$ are closed and disjoint, then there are disjoint open subsets G_1, $G_2 \subset X$ with $F_1 \subset G_1$ and $F_2 \subset G_2$. Thus, we say that a compact Hausdorff topology not only separates points, but also separates sets.

Email comments and suggestions to `rbdt2@cam.ac.uk`