ANALYSIS AND TOPOLOGY—EXAMPLES 2

(updated 27 September 2024)

Exercises

1. Let (X, d) and (Y, d') be metric spaces, and $f\colon X \to Y$ be a map. Recall from lectures the following definition of continuity: we say f is continuous at $x \in X$ if

$$\forall \epsilon > 0, \exists \delta = \delta(x, \epsilon) > 0 \text{ such that } d(x, y) < \delta \implies d'(f(x), f(y) < \epsilon.$$

(a) Show that the following is an equivalent definition: f is continuous at x if $\forall V \subset Y$ which is a neighborhood of $f(x)$, $U = f^{-1}(V)$ is a neighborhood of x. This second, more abstract, definition will come in handy soon!

(b) Deduce that f is continuous in X if and only if $\forall V \subset Y$ open in Y, $f^{-1}(V)$ is open in X.

(c) Show that f is continuous in X if and only if $\forall V \subset Y$ closed in Y, $f^{-1}(V)$ is closed in X.

2. Consider a sequence (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, such that $f_n \rightrightarrows f$ uniformly.

(a) Show that f must be uniformly continuous. Thus, uniform limits preserve uniform continuity.

(b) Is the conclusion still true if we only assume $f_n \to f$ pointwise? Give a proof or counterexample.

3. Consider $\mathbb{Q} \cap [0, 1]$ with the usual Euclidean metric. Show that it is totally bounded. Show, using the definition of compactness, that it is also not compact. Is this in contradiction with the Heine–Borel theorem from lectures?

4. The goal of this question is to reprove Heine–Borel and Heine–Cantor theorems in the easier Euclidean setting.

(a) Show, without appealing to the Heine–Borel theorem for general metric spaces from lectures, that $K \subset \mathbb{R}^n$ is compact iff it is closed and bounded. You may use the fact that, for a metric space, closure/compactness is the same as sequential closure/compactness, respectively.

(b) Show, without appealing to the Heine–Cantor theorem for general metric spaces from lectures, that if $f\colon [a, b] \to \mathbb{R}$ is continuous it is also uniformly continuous. You may use the fact that, for a metric space, continuity is the same as sequential continuity.

Problems

5. For each $k \in \mathbb{N}_0$, define the metric space $C^k([a, b])$ as the space of k times continuously differentiable real-valued functions on $[a, b]$ endowed with the metric

$$d_{C^k}(f, g) = \sum_{j=0}^{k} \sup_{x \in [a,b]} \left| \frac{d^j f}{dx^j}(x) - \frac{d^j g}{dx^j}(x) \right|.$$

(a) Show that $C^k([a, b])$ is a complete metric space.

(b) Show that the map $f \mapsto \int_a^x f(y)dy$ is a continuous map from $C^k([a, b])$ to $C^{k+1}([a, b])$.

(c) Show that the map $f \mapsto f'$ is a continuous map from $C^{k+1}([a, b])$ to $C^k([a, b])$.

6. Which of the following functions $f\colon [0, \infty) \to \mathbb{R}$ are uniformly continuous?

(a) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$;
(b) $f(x) = \sin x^2$;
(c) $f(x) = (\sin x^3)/(x^2 + 1)$.

7. Let $f\colon [0, \infty) \to [0, \infty)$ be uniformly continuous and such that $\int_0^\infty f(x)dx$ exists and is finite.

(a) Show that $\lim_{x\to\infty} f(x) = 0$.
(b) Is this still true if f is only continuous? Give a proof or counterexample.

8. In this question we consider spaces of real sequences.

(a) Justify that the metric space ℓ_∞, of bounded real sequences and equipped with metric $d((x_n), (y_n)) = \sup_{n\in\mathbb{N}} |x_n - y_n|$, is complete.
(b) Consider the space $\ell_2 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} | \sum x_n^2 \text{ converges}\}$ with

$$d((x_n), (y_n)) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Show that d is a well-defined metric on ℓ_2 and that (ℓ_2, d) is complete.

9. On a non-empty complete metric space (X, d), consider a continuous map $f\colon X \to X$ with the property that for each pair $x, y \in X$ with $x \neq y$, there is a number $L \in (0, 1)$ (which depends on the pair x, y) such that $d(f(x), f(y)) \leq Ld(x, y)$.

(a) Show that, if f has a fixed point, then it is unique.
(b) Must f be a contraction? Must f have a fixed point? Give a proof or counterexample.
(c) Show that, if (X, d) is assumed to be compact, then f necessarily has a unique fixed point.

10. Take $0 < a, b < \infty$. Let $f\colon [-a, a] \times [-b, b] \to \mathbb{R}$ be a continuous function. Consider the initial value problem

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(0) = y_0 \end{cases}.$$

(a) Is the solution necessarily unique? Give a proof or counterexample.
(b) Show that for every $k \in \mathbb{N}$, there is sequence of intervals I_k, which are neighborhood of zero, and functions $y_k \in C(I_k)$ solving the initial value problem

$$\begin{cases} y_k'(x) = f_k(x, y_k(x)), \\ y_k(0) = y_0 \end{cases}.$$

where (f_k) are Lipschitz functions satisfying $f_k \rightrightarrows f$ on $[-a, a] \times [b, b]$. *Hint: you may assume, without proof, the statement of problem 12 below.* What happens as $k \to \infty$?
(c) ($\star$) Show that there is a solution $y(x)$ for $x \in (-\epsilon, \epsilon)$, where $0 < \epsilon \leq a$. *Hint: you may assume, without proof, the statements in problems 12 and 13(d) below.* Compare with your answer in part (b).

11. On a normed vector space there is a canonical choice of metric which makes it a metric space: $d(x, y) = \|x-y\|$. Show that on a finite-dimensional real vector space, every choice of norm (and thus canonical metric) is equivalent. Is this still true for infinite-dimensional vector spaces?

OPTIONAL extra problems (not for marking)

12. The goal of this question is to guide you through the proof of the Stone–Weierstrass theorem, which asserts that a continuous function $f\colon K \to \mathbb{R}$, where $K = [a_1, b_1] \times \cdots \times$

$[a_n, b_n] \subset \mathbb{R}^n$, can be approximated uniformly by a sequence of polynomials. (Compare with Taylor series: we don't require analyticity here for the convergence to occur.) **We take $n = 1$ in what follows.**

(a) Argue that, without loss of generality, it is enough to show the statement with $[a_1, b_1] = [0, 1]$.

(b) Letting $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, show that one has

$$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x) = \frac{1}{n} x(1-x).$$

(c) Now let $p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x)$. Use the first identity to show

$$|p_n(x) - f(x)| \leq \sum_{|x-\frac{k}{n}|<\delta} |f(k/n) - f(x)|\, p_{k,n}(x) + \sum_{|x-\frac{k}{n}|\geq\delta} |f(k/n) - f(x)|\, p_{k,n}(x),$$

for any $\delta > 0$.

(d) Choose an appropriate δ, and use the second identity for $p_{n,k}$, to conclude that $p_n \rightrightarrows f$ in $[0, 1]$.

13. Let (X, d) be compact metric space, and let $C(X)$ denote the space of all real-valued continuous functions on X. We say that $F \subset C(X)$ is equicontinuous if every $x \in X$ and every $\epsilon > 0$, x has a neighborhood U_x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U_x$ and all $f \in F$. (Note: the neighborhood U_x is the same for the entire family of functions F.) The goal of this question is to show that $F \subset C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.

(a) Using results from the lectures, justify that it is enough to show $F \subset C(X)$ is totally bounded if and only if it is bounded and equicontinuous.

(b) Show the ($\Rightarrow$) direction: if $F \subset C(X)$ is totally bounded, then it is bounded and equicontinuous.

(c) ($\star$) Show the ($\Leftarrow$) direction. *Hint: use the compactness of X to construct a finite subfamily of F that is a candidate ϵ-net for F.*

(d) Deduce that if (f_n) is a family of real-valued continuous functions on a closed and bounded interval $[a, b]$ of the real line, which is bounded and equicontinuous relative to the uniform metric (equivalently, bounded and equicontinuous uniformly in n), then there is a subsequence (f_{n_k}) that converges uniformly. Compare with question 13 from sheet 1.