ANALYSIS AND TOPOLOGY—EXAMPLES 1

(updated 26 September 2024)

Exercises

1. Let (X,d) be a metric space.

(a) Show that $U \subset X$ is open in X if and only if U is a neighborhood for all of its points.

(b) Show that $A \subset X$ is closed in X if and only if $A = \{x \in X\colon \inf_{y\in A} d(x,y) = 0\}$.

2. Let (X,d) be a metric space. In the definition of the topology of (X,d), why don't we consider infinite intersections of open sets to be open? *Hint: give a counterexample with* $X = \mathbb{R}$.

3. Let (X,d) be a metric space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. In lectures, we said that x_n converges to $x \in X$ if

$$\forall \epsilon > 0\, \exists N = N(\epsilon) > 0 \text{ s.t. } d(x_n, x) < \epsilon \text{ for all } n \geq N. \quad (*)$$

(a) Show that $x_n \to x$ equivalent to the following statement:

$$\forall V \subset X \text{ nhood of } x\ \exists N = N(V) > 0 \text{ s.t. } x_n \in V \text{ for all } n \geq N. \qquad (**)$$

(b) In lectures, we used $(**)$ to show that if $\{x_n\}_{n=1}^{\infty}$ converges in X, then its limit is unique. Repeat the proof with definition $(*)$.

4. Let (f_n) and (g_n) be sequences of real-valued functions on a set X converging uniformly: $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$.

(a) Show that $f_n + g_n \rightrightarrows f + g$.

(b) Show that $f_n g_n \to fg$ pointwise, but that $f_n g_n$ does not converge uniformly to fg in general. If f and g are bounded, does the latter conclusion change? What if f is bounded but g is not?

5. Let (f_n) be a sequence of continuous functions on $[0,1]^k \subset \mathbb{R}^k$, and suppose that $f_n \rightrightarrows f$ uniformly. Let (x_m) be a convergent sequence of points in $[0,1]^k$, and suppose that $x_m \to x$.

(a) Show that if $f_n \rightrightarrows f$ uniformly, then $f_n(x_n) \to f(x)$. Thus, we say uniform convergence preserves sequential continuity.

(b) Is (a) still true if f_n converges pointwise, but not necessarily uniformly, to f?

Problems

6. For each of the following sets X, determine whether or not the given function d defines a metric on X.

(a) $X = \mathbb{R}^n$; $d(x,y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.

(b) $X = \mathbb{C}$;

$$d(z,w) = \begin{cases} |z - w|, & \text{if } z \text{ and } w \text{ lie on the same line through the origin,} \\ |z| + |w|, & \text{otherwise.} \end{cases}$$

(c) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$, i.e. sequences with values in $\mathbb{N}$; for N the smallest natural number such that $x_n \neq y_n$,

$$d(x_n, y_n) = \begin{cases} 0, & \text{if } x_n = y_n\ \forall n \in \mathbb{N}, \\ 2^{-N}, & \text{otherwise.} \end{cases}$$

(d) X is the set of closed, non-empty, bounded subsets of a metric space (M, d');

$$d(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} d'(a,b), \sup_{b\in B}\inf_{a\in A} d'(a,b)\}, \qquad A, B \in X.$$

Note: we say a subset U of a metric space (M, d') is bounded if there exists $r > 0$ such that $d'(x,y) < r$ for all $x, y \in U$.

7. For an alphabetic string s of length $L \in \mathbb{N}$, let $s[i]$ be the character at position i, for $i = 1, \dots, L$. Let X_L be the set of strings of length L. In computer science, one sometimes considers the functions on X_L given by

(a) $d_1(s, s') = \#\{i \colon s[i] \neq s'[i]\}$;

(b) $d_2(s, s')$ is the minimum number of single-character insertions, deletions or substitutions required to change s into s'.

Show that both are metrics on X_L. For which $L \in \mathbb{N}$ are they equivalent metrics?

8. Consider the following subsets of $\mathbb{R}$ with the Euclidean metric. Determine if they are open or closed.

(a) $U = (0,1) \cup (2,3)$.

(b) $U = [0,1)$. What if we replace the metric space $\mathbb{R}$ by the metric space $[-1,1]$, the metric space $[0,1)$ or the metric space $[0,2]$, in each case with the subspace metric inherited from the Euclidean metric?

(c) $U = \cup_{n=1}^{\infty} (-n, \sin(n))$.

9. Consider the following subsets of $\mathbb{R}^2$ with the Euclidean metric. Determine if they are open or closed.

(a) $\{(x,y) : y > 0\}$;

(b) $\{(x,y) : y \neq 0\}$;

(c) $\{(x,0) : 0 \leq x \leq 1\}$;

(d) $\{(x,0) : 0 < x < 1\}$;

(e) $\{(x,y) : y/x \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;

(f) $\{(x,y) : y/x \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;

10. Consider the following subsets of $C([0,1])$ with the uniform metric $d(f,g) = \sup_{[0,1]} |f - g|$. Determine if they are open or closed.

(a) $\{f \in C([0,1]) \mid f(1/2) = 0\}$;

(b) $\{f \in C([0,1]) \mid \int_0^1 f = 0\}$.

Does the answer change if we replace the uniform metric with the L^1 metric $d(f,g) = \int_0^1 |f - g|$?

11. ($\star$) Show that the function $1_{\mathbb{Q}}$ is not the pointwise limit of any sequence of continuous functions.

12. Which of the following sequences (f_n) of functions converge uniformly on the set X? If it exists, what is their uniform limit?

(a) $f_n(x) = x^{2n}$ on $X = (0,1)$;

(b) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;

(c) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

13. ($\star$) Let (f_n) be a sequence of real-valued functions on a set X, which are pointwise bounded: $\sup_{n\in\mathbb{N}} |f_n(x)| \leq g(x) < \infty$ for a function g on X.

(a) Let $X = \mathbb{N}$, show that (f_n) has a convergent subsequence.

(b) If $X = \mathbb{R}$, does (f_n) still have a convergent subsequence?

14. Prove that a bounded sequence in $\mathbb{R}^n$ must have a convergent subsequence. *Hint: in Analysis I you saw this is true for $n = 1$; try a proof by induction on n.*

Email comments and suggestions to `rbdt2@cam.ac.uk`