1. Let $X = [0,1] \cup [2,3]$ with the usual topology and define an equivalence relation $\sim$ on X by $x \sim y$ iff $x = y$ or $\{x, y\} = \{1, 2\}$. Show that $X/\sim$ is homeomorphic to $[0,1]$ with the usual topology.

2. Let X, Y be topological spaces and endow $X \times Y$ with the product topology.

(a) Show for each $y \in Y$ that $X \times \{y\}$, as a subspace of $X \times Y$, is homeomorphic to X.

(b) Define an equivalence relation $\sim$ on $X \times Y$ by $(x, y) \sim (x', y')$ iff $x = x'$. Assuming $Y \neq \emptyset$, show that $X \times Y/\sim$ is homeomorphic to X.

3. Let (X, d) and (Y, e) be metric spaces. Let τ, σ be the topologies induced by d, e respectively. Define $f\colon (X \times Y)^2 \to \mathbb{R}$ by $f((x, y), (z, w)) = \max\{d(x, z), e(y, w)\}$. Show that f is a metric on $X \times Y$ and that the topology it induces is the product topology induced by the topologies τ on X and σ on Y. Deduce that if $\mathbb{R}$ has the usual topology then the product topology on $\mathbb{R} \times \mathbb{R}$ is exactly the Euclidean topology.

4. Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(x, y) \sim (z, w)$ iff $x - z \in \mathbb{Z}$ and $y - w \in \mathbb{Z}$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to the torus

$$T = \left\{ \left((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta\right) \middle| \theta, \phi \in [0, 2\pi] \right\}.$$

5. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df|_A$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2 f|_A$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

6. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

7. At which points is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |x||y|$ differentiable? What about the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} xy/\sqrt{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}?$$

8. Let $\mathcal{N}_n \subset \mathcal{M}_n$ be the set of invertible $n \times n$ matrices. Show that $\mathcal{N}_n$ is an open subset of $\mathcal{M}_n$.

Define $f: \mathcal{N}_n \to \mathcal{N}_n$ by $f(A) = A^{-1}$. Show that f is differentiable at the identity matrix I, and that $Df|_I(H) = -H$.

Let $A \in \mathcal{N}_n$. By writing $(A+H)^{-1} = A^{-1}(I + HA^{-1})^{-1}$, or otherwise, show that f is differentiable at A. What is $Df|_A$?

Show further that f is twice-differentiable at I, and find $D^2 f|_I$ as a bilinear map.

9. Recall that the function $\det: \mathcal{M}_n \to \mathbb{R}$ is differentiable at every invertible matrix A with $D\det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Show that det is twice differentiable at I and find $D^2 \det|_I$ as a bilinear map.

10. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a, b), and that $D_2 f$ exists at (a, b). Show that f is differentiable at (a, b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a, b), and that for fixed x the function $y \mapsto f(x, y)$ is continuous. Show that f is continuous at (a, b).

11. Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g: B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

12. Consider the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, x^3 + y^3 - 3xy)$ and the set $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that f is locally invertible around each point of C except $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \setminus \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $f(x_0, y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0, y_0) \in C$ other than $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x, y) \in C$.