There is no question 3 on this sheet. The original question 3 was a duplicate of a question on sheet 2 so I deleted it, but I've not changed the numbering so that it's consistent between the online and hard copy versions.

1. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean metric are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \neq 0\}$;
(iv) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(v) $\{(x,y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

2. Is the set $(1,2]$ an open subset of the metric space $\mathbb{R}$ with the usual metric? Is it closed? What if we replace the metric space $\mathbb{R}$ by the metric space $[0,2]$, the metric space $(1,3)$ or the metric space $(1,2]$, in each case with the usual metric?

4. Let W, X and Y be topological spaces, let $f\colon W \to X$, let $g\colon X \to Y$ and let $a \in W$. Suppose f is continuous at a and g is continuous at $f(a)$. Show that $g \circ f$ is continuous at a.

5. Let X be a Hausdorff space and let $\Delta = \{(x,x) | x \in X\}$. Show that Δ is a closed subset of $X \times X$ with the product topology.

6. Let X, Y be topological spaces with Y Hausdorff and let f, $g\colon X \to Y$ be continuous. Show that $\{x \in X | f(x) = g(x)\}$ is a closed subset of X.

7. Is a topological space with the cofinite topology compact? What about the cocountable topology?

8. Let X be a compact Hausdorff space and let F_1, $F_2 \subset X$ be closed and disjoint. Show that there are disjoint open subsets G_1, $G_2 \subset X$ with $F_1 \subset G_1$ and $F_2 \subset G_2$.

9. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean topology are connected? Which are path-connected? (And why?)

(i) $\{(x,y) \in \mathbb{R}^2 : \|(x,y)-(-1,0)\| \leqslant 1 \text{ or } \|(x,y)-(1,0)\| < 1\}$;

(ii) $\{(x,y) \in \mathbb{R}^2 | x = 0 \text{ or } y/x \in \mathbb{Q}\}$;

(iii) $\{(x,y) \in \mathbb{R}^2 | x = 0 \text{ or } y/x \in \mathbb{Q}\} \backslash \{(0,0)\}$.

10. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a decreasing sequence of connected, compact subsets of a Hausdorff space X. Show that $\bigcap_{n=1}^{\infty} K_n$ is connected. Give an example with $X = \mathbb{R}^2$ to show that this need not be true if we replace 'compact' with 'closed'.

11. Find the connected components of the space $X = \{(0,0),(0,1)\} \cup (\{1/n | n \in \mathbb{N}\} \times [0,1])$ as a subspace of $\mathbb{R}^2$ with the Euclidean topology. Show that there are points x, $y \in X$ belonging to different components of X but such that we cannot find disjoint, open U, $V \subset X$ with $x \in U$, $y \in V$ and $U \cup V = X$.

12. Let X be a topological space. The *interior* of a set $A \subset X$ is the largest open set A° contained in A, and the *closure* of A is the smallest closed set $\bar{A}$ containing A.

(a) Why do these definitions make sense?

(b) Show that, starting from any set A, we cannot obtain more than seven distinct sets by repeatedly applying the operations of interior and closure.

(c) Give an example of a subset $A \subset \mathbb{R}$ where we obtain exactly seven distinct sets by this procedure.

$^+$13. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function under which the image of each path-connected set is path-connected and the image of each compact set is compact. Show that f is continuous.