1. Show that $7^n \to 0$ in $\mathbb{Z}$ with the 7-adic metric.

2. Let (W, c), (X, d) and (Y, e) be metric spaces, let $f\colon W \to X$, let $g\colon X \to Y$ and let $a \in W$. Suppose f is continuous at a and g is continuous at $f(a)$. Show directly from the definition of continuity that $g \circ f$ is continuous at a.

3. For each of the following sets X, determine whether or not the given function d defines a metric on X.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

4. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X + Y$ is not closed. Show that it is not possible to find such an example with X bounded.

5. Is the set $\{f \in C([0, 1]) \mid f(1/2) = 0\}$ closed in the space $C([0, 1])$ with the uniform metric? What about the set $\{f \in C([0, 1]) \mid \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform metric with the L_1 metric $d(f, g) = \int_0^1 |f - g|$?

6. (a) Explain briefly why the metric space ℓ_∞ of bounded real sequences with metric $d((x_n), (y_n)) = \sup_{n \in \mathbb{N}} |x_n - y_n|$ is complete.

(b) Let $\ell_2 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid \sum x_n^2 \text{ converges}\}$ with metric $d((x_n), (y_n)) = \sqrt{\sum_{n=1}^\infty (x_n - y_n)^2}$. Prove carefully that d is indeed a well-defined metric on ℓ_2 and that (ℓ_2, d) is complete. Is (ℓ_2, d) sequentially compact?

7. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

8. Give an example of a non-empty complete metric space (X, d) and a function $f: X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is sequentially compact. Show that in this case f must have a fixed point. If $g: X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all x, $y \in X$, must g have a fixed point?

9. Let $a \in (0, \infty)$ and let $X = C([0, a])$. Let $\alpha \in \mathbb{R}$. Show that we can define a metric d_α on X by $d_\alpha(f, g) = \sup_{x \in [0,a]} |f(x) - g(x)| e^{-\alpha x}$. Show that d_α is equivalent to the uniform metric d, and moreover that a sequence in X is Cauchy under the metric d_α if and only if it is Cauchy under the metric d. Deduce that (X, d_α) is a complete metric space.

Now let $\phi: \mathbb{R}^2 \to \mathbb{R}$ be continuous and suppose that there exists some $K > 0$ such that for all $t \in \mathbb{R}$ and all x, $y \in \mathbb{R}$ we have $|\phi(t, x) - \phi(t, y)| \leqslant K|x - y|$. Let $y_0 \in \mathbb{R}$ and define $T: X \to X$ by $Tf(x) = y_0 + \int_0^x \phi(t, f(t))\, dt$ for all $f \in X$ and all $x \in [0, a]$. Give an example to show that T need not be a contraction under the uniform metric. Show, however, that T is a contraction under the metric d_α for some value of α. Deduce that the initial value problem $f'(x) = \phi(x, f(x))$ with $f(0) = y_0$ has a unique solution on the interval $[0, a]$.

10. Which functions $f: \mathbb{R} \to X$ are continuous, where $\mathbb{R}$ has the usual metric and X is a discrete metric space?

11. Let (X, d) be a non-empty complete metric space and let $f: X \to X$ be a function such that for each positive integer n we have

(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and

(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.

Must f have a fixed point?

12. Let (X, d) be a non-empty complete metric space, let $f: X \to X$ be a continuous function, and let $K \in [0, 1)$.

(a) Suppose we assume that for all x, $y \in X$ we have either $d(f(x), f(y)) \leq Kd(x, y)$ or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$. Show that f has a fixed point.

$^+$(b) Suppose instead we only assume that for each x, $y \in X$ at least on of the following holds: $d(f(x), f(y)) \leq Kd(x, y)$, or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$ or $d(f(f(f(x))), f(f(f(y)))) \leq Kd(x, y)$. Must f have a fixed point?