1. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df|_A$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2 f|_A$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

2. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

3. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

4. At which points is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |x||y|$ differentiable? What about the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} xy/\sqrt{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}?$$

5. Let $\mathcal{N}_n \subset \mathcal{M}_n$ be the set of invertible $n \times n$ matrices. Show that $\mathcal{N}_n$ is an open subset of $\mathcal{M}_n$.

Define $f\colon \mathcal{N}_n \to \mathcal{N}_n$ by $f(A) = A^{-1}$. Show that f is differentiable at the identity matrix I, and that $Df|_I(H) = -H$.

Let $A \in \mathcal{N}_n$. By writing $(A + H)^{-1} = A^{-1}(I + HA^{-1})^{-1}$, or otherwise, show that f is differentiable at A. What is $Df|_A$?

Show further that f is twice-differentiable at I, and find $D^2 f|_I$ as a bilinear map.

6. Recall that the function $\det\colon \mathcal{M}_n \to \mathbb{R}$ is differentiable at every invertible matrix A with $D\det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Show that det is twice differentiable at I and find $D^2 \det|_I$ as a bilinear map.

7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a, b), and that $D_2 f$ exists at (a, b). Show that f is differentiable at (a, b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a, b), and that for fixed x the function $y \mapsto f(x, y)$ is continuous. Show that f is continuous at (a, b).

8. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that

(i) f is continuous on $\mathbb{R}^2$;

(ii) the partial derivatives $D_1 f$ and $D_2 f$ exist and are continuous on $\mathbb{R}^2$; and

(iii) the partial derivatives $D_1 D_2 f$ and $D_2 D_1 f$ exist on $\mathbb{R}^2$.

Where are $D_1 D_2 f$ and $D_2 D_1 f$ continuous? Is $D_1 D_2 f(0, 0) = D_2 D_1 f(0, 0)$?

9. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

10. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, x^3 + y^3 - 3xy)$ and the set $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that f is locally invertible around each point of C except $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \backslash \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $f(x_0, y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0, y_0) \in C$ other than $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x, y) \in C$.