1. Let W, X and Y be topological spaces, let $f\colon W \to X$, let $g\colon X \to Y$ and let $a \in W$. Suppose f is continuous at a and g is continuous at $f(a)$. Show that $g \circ f$ is continuous at a.

2. Let X be a Hausdorff space and let $\Delta = \{(x,x) | x \in X\}$. Show that Δ is a closed subset of $X \times X$ with the product topology.

3. Let X, Y be topological spaces with Y Hausdorff and let f, $g\colon X \to Y$ be continuous. Show that $\{x \in X | f(x) = g(x)\}$ is a closed subset of X.

4. Is a topological space with the cofinite topology compact? What about the cocountable topology?

5. Let $X = [0.1] \cup [2,3]$ with the usual topology and define an equivalence relation $\sim$ on X by $x \sim y$ iff $x = y$ or $\{x,y\} = \{1,2\}$. Show that $X/\sim$ is homeomorphic to $[0,1]$ with the usual topology.

6. Let X, Y be topological spaces and endow $X \times Y$ with the product topology.

(a) Show for each $y \in Y$ that $X \times \{y\}$, as a subspace of $X \times Y$, is homeomorphic to X.

(b) Define an equivalence relation $\sim$ on $X \times Y$ by $(x,y) \sim (x',y')$ iff $x = x'$. Assuming $Y \neq \emptyset$, show that $X \times Y/\sim$ is homeomorphic to X.

7. Let X be a compact Hausdorff space and let F_1, $F_2 \subset X$ be closed and disjoint. Show that there are disjoint open subsets G_1, $G_2 \subset X$ with $F_1 \subset G_1$ and $F_2 \subset G_2$.

8. Let (X,d) and (Y,e) be metric spaces. Let τ, σ be the topologies induced by d, e respectively. Define $f : (X \times Y)^2 \to \mathbb{R}$ by $f((x,y),(z,w)) = \max\{d(x,z), e(y,w)\}$. Show that f is a metric on $X \times Y$ and that the topology it induces is that generated by the π-system $\tau \times \sigma$. Deduce that if $\mathbb{R}$ has the usual topology then the product topology on $\mathbb{R} \times \mathbb{R}$ is exactly the Euclidean topology.

9. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean topology are connected? Which are path-connected? (And why?)

(i) $\{(x,y) \in \mathbb{R}^2 : \|(x,y) - (-1,0)\| \leqslant 1 \text{ or } \|(x,y) - (1,0)\| < 1\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 | x = 0 \text{ or } y/x \in \mathbb{Q}\}$;

(iii) $\{(x, y) \in \mathbb{R}^2 | x = 0 \text{ or } y/x \in \mathbb{Q}\} \setminus \{(0, 0)\}$.

10. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a decreasing sequence of connected, compact subsets of a Hausdorff space X. Show that $\bigcap_{n=1}^{\infty} K_n$ is connected. Give an example with $X = \mathbb{R}^2$ to show that this need not be true if we replace 'compact' with 'closed'.

11. Find the connected components of the space $X = \{(0, 0), (0, 1)\} \cup (\{1/n | n \in \mathbb{N}\} \times [0, 1])$ as a subspace of $\mathbb{R}^2$ with the Euclidean topology. Show that there are points x, $y \in X$ belonging to different components of X but such that we cannot find disjoint, open U, $V \subset X$ with $x \in U$, $y \in V$ and $U \cup V = X$.

12. Let X be a topological space. The *interior* of a set $A \subset X$ is the largest open set A° contained in A, and the *closure* of A is the smallest closed set $\bar{A}$ containing A.

(a) Why do these definitions make sense?

(b) Show that, starting from any set A, we cannot obtain more than seven distinct sets by repeatedly applying the operations of interior and closure.

(c) Give an example of a subset $A \subset \mathbb{R}$ where we obtain exactly seven distinct sets by this procedure.

$^+$13. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function under which the image of each path-connected set is path-connected and the image of each compact set is compact. Show that f is continuous.