1. Show that $7^n \to 0$ in $\mathbb{Z}$ with the 7-adic metric.

2. Let (W, c), (X, d) and (Y, e) be metric spaces, let $f \colon W \to X$, let $g \colon X \to Y$ and let $a \in W$. Suppose f is continuous at a and g is continuous at $f(a)$. Show directly from the definition of continuity that $g \circ f$ is continuous at a.

3. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean metric are open? Which are closed? (And why?)

(i) $\{(x, 0) : 0 \leq x \leq 1\}$;
(ii) $\{(x, 0) : 0 < x < 1\}$;
(iii) $\{(x, y) : y \neq 0\}$;
(iv) $\{(x, y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x, y) : x = 0\}$;
(v) $\{(x, y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x, y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

4. Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with the usual metric? Is it closed? What if we replace the metric space $\mathbb{R}$ by the metric space $[0, 2]$, the metric space $(1, 3)$ or the metric space $(1, 2]$, in each case with the usual metric?

5. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.
(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.
(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.
(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

6. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X + Y$ is not closed. Show that it is not possible to find such an example with X bounded. If V, $W \subset \mathbb{R}^n$ are open, must $V + W$ be open?

7. Is the set $\{f \in C([0, 1]) \mid f(1/2) = 0\}$ closed in the space $C([0, 1])$ with the uniform metric? What about the set $\{f \in C([0, 1]) \mid \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform metric with the L_1 metric $d(f, g) = \int_0^1 |f - g|$?

8. (a) Explain briefly why the metric space ℓ_∞ of bounded real sequences with metric $d((x_n),(y_n)) = \sup_{n\in\mathbb{N}} |x_n - y_n|$ is complete.

(b) Let $\ell_2 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} | \sum x_n^2 \text{ converges}\}$ with metric

$$d((x_n),(y_n)) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Prove carefully that d is indeed a well-defined metric on ℓ_2 and that (ℓ_2, d) is complete. Is (ℓ_2, d) compact?

9. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

10. Give an example of a non-empty complete metric space (X, d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is compact. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all x, $y \in X$, must g have a fixed point?

11. Let (X, d) be a compact metric space and $f\colon X \to X$. Suppose $d(f(x), f(y)) = d(x, y)$ for all x, $y \in X$. Show that f is bijective.

12. Which functions $f\colon \mathbb{R} \to X$ are continuous, where $\mathbb{R}$ has the usual metric and X is a discrete metric space?

13. Let (X, d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have

(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and

(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.

Must f have a fixed point?

14. Let (X, d) be a non-empty complete metric space, let $f\colon X \to X$ be a continuous function, and let $K \in [0, 1)$.

(a) Suppose we assume that for all x, $y \in X$ we have either $d(f(x), f(y)) \leq Kd(x, y)$ or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$. Show that f has a fixed point.

$^+$(b) Suppose instead we only assume that for each x, $y \in X$ at least on of the following holds: $d(f(x), f(y)) \leq Kd(x, y)$, or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$ or $d(f(f(f(x))), f(f(f(y)))) \leq Kd(x, y)$. Must f have a fixed point?