Mich. 2022 **ANALYSIS AND TOPOLOGY—EXAMPLES 1** PAR

1. Let (z_n) be a sequence in $\mathbb{R}^2$ such that $z_n \to a$ and $z_n \to b$. Show that $a = b$.

2. Let (z_n) and (w_n) be sequences in $\mathbb{R}^2$ with $z_n \to z$ and $w_n \to w$. Show that the scalar product $z_n \cdot w_n \to z \cdot w$.

3. Let $X \subset \mathbb{R}^2$ and $f\colon X \to \mathbb{R}$. We say X is *closed* if whenever (z_n) is a sequence in X with $z_n \to z \in \mathbb{R}^2$ then $z \in X$. We say X is *bounded* if there is some $M \in \mathbb{R}$ such that for all $z \in X$ we have $\|z\| \leqslant M$. We say f is *continuous* if for all $z \in X$ and all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $y \in X$ with $\|y - z\| < \delta$ we have $|f(y) - f(z)| < \varepsilon$. Show that if X is closed and bounded and f is continuous then $f(X) \subset \mathbb{R}$ is bounded and there are z_1, $z_2 \in X$ such that $f(z_1) = \inf f(X)$ and $f(z_2) = \sup f(X)$.

4. Prove, by induction on n or otherwise, that a bounded sequence in $\mathbb{R}^n$ must have a convergent subsequence. Prove also that any Cauchy sequence in $\mathbb{R}^n$ must be convergent.

5. Which of the following sequences (f_n) of functions converge uniformly on the set X?
(a) $f_n(x) = x^n$ on $X = (0, 1)$; (b) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
(c) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

6. Construct a sequence (f_n) of *continuous* real-valued functions on $[-1, 1]$ converging pointwise to the zero function but with $\int_{-1}^{1} f_n \not\to 0$. $^+$Is it possible to find such a sequence with $|f_n(x)| \leq 1$ for all n and for all x?

7. Let (f_n) and (g_n) be sequences of real-valued functions on a subset X of $\mathbb{R}$ converging uniformly to f and g respectively. Show that the pointwise sum $f_n + g_n$ converges uniformly to $f + g$. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg, but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

8. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_m) is a sequence of points in $[a, b]$ with $x_m \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_m \to x$ in $[a, b]$ such that $f_n(x_n) \not\to f(x)$.

9. Which of the following functions $f\colon [0, \infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?
(i)$f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x + 1)$.

10. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

11. Let $X \subset \mathbb{R}^2$ and $f\colon X \to \mathbb{R}^2$. Write down sensible definitions of what it should mean for f to be *continuous* and for f to be *uniformly continuous*. Show that if X is closed and bounded and f is continuous then f must be uniformly continuous.

12. Show that, for any $x \in X = \mathbb{R} - \{1, 2, 3, \dots\}$, the series $\sum_{m=1}^{\infty} (x - m)^{-2}$ converges. Define $f\colon X \to \mathbb{R}$ by $f(x) = \sum_{m=1}^{\infty} (x - m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n\colon X \to \mathbb{R}$ by $f_n(x) = \sum_{m=1}^{n} (x - m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X? Is f continuous?

13. Let $f_n\colon \mathbb{N} \to \mathbb{R}$ for each $n \geqslant 1$. Suppose (f_n) is pointwise bounded. Must it have a pointwise convergent subsequence? What if we replace $\mathbb{N}$ with $\mathbb{R}$?

14. Construct a function $f\colon [0, 1] \to \mathbb{R}$ which is not the pointwise limit of any sequence of continuous functions.

15. Let (f_n) be a pointwise bounded sequence of continuous, real-valued functions on $[0, 1]$. Show that there is some subinterval $[a, b]$ of $[0, 1]$ with $a < b$ on which (f_n) is uniformly bounded.