1. Prove the following statements.

(i) In a topological space, every subsequence of a convergent sequence is convergent.

(ii) In a metric space, a Cauchy sequence with a convergent subsequence is convergent.

2. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)

(i) $\{(x,0) : 0 \leqslant x \leqslant 1\}$;

(ii) $\{(x,0) : 0 < x < 1\}$;

(iii) $\{(x,y) : y \neq 0\}$;

(iv) $\bigcup_{n\in\mathbb{N}}\{(x,y) : y = nx\} \cup \{(0,y) : y \in \mathbb{R}\}$;

(v) $\bigcup_{q\in\mathbb{Q}}\{(x,y) : y = qx\} \cup \{(0,y) : y \in \mathbb{R}\}$;

(vi) $\{(x,f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

3. Is the set $(1,2]$ an open subset of $\mathbb{R}$? Is it closed? What if we replace $\mathbb{R}$ with the subspace $(1,3)$, $[0,2]$ or $(1,2]$?

4. Let M, N be metric spaces, and let $A \subset M$ and $B \subset N$. Show that if both A and B are open/closed/bounded, then so is $A \times B$ in the product metric space $M \oplus_\infty N$.

5. Let M be a metric space and $A \subset M$. Prove that

$$\overline{A} = \{x \in M : \exists\,(x_n) \text{ in } A \text{ such that } x_n \to x\} \quad \text{and} \quad A^\circ = \{x \in M : \exists\, r > 0\ D_r(x) \subset A\}\ .$$

Show further that if A is non-empty, then $x \mapsto d(x,A) = \inf\{d(x,a) : a \in A\}$ is a 1-Lipschitz map $M \to \mathbb{R}$ and that $\overline{A} = \{x \in M : d(x,A) = 0\}$. Deduce, or otherwise show, that if A and B are disjoint closed subsets of M, then there exist disjoint open subsets U and V of M such that $A \subset U$ and $B \subset V$.

Finally, show that the inclusions $D_r(x) \subset B_r(x)^\circ$ and $\overline{D_r(x)} \subset B_r(x)$ hold in every metric space and can be strict in general but that they are always equalities in $\mathbb{R}^n$.

6. Which of the following metric spaces are complete?

(i) $C^1[0,1] = \{f\colon [0,1] \to \mathbb{R} : f \text{ continuously differentiable}\}$ in the uniform metric D;

(ii) $C^1[0,1]$ in the metric $D'(f,g) = D(f,g) + D(f',g')$;

(iii) $C[0,1]$ in the L_1-metric.

(iv) $\mathbb{Z}$ in the 2-adic metric.

7. Let X be an uncountable set. Show that the family τ consisting of the emptyset and all subsets of X with countable complement is a topology on X. Fix $x \in X$ and let $Y = X \setminus \{x\}$. Show that Y is dense in X but no sequence in Y converges to x. Is the space (X,τ) metrizable? Identify the convergent sequences in X and the continuous functions $X \to \mathbb{R}$.

8. The diagonal of a set Y is the set $\Delta_Y = \{(x,y) \in Y \times Y : x = y\}$. Show that a topological space Y is Hausdorff if and only if Δ_Y is closed in $Y \times Y$. Deduce or otherwise show that if $f, g \colon X \to Y$ are continuous functions from a space X to a Hausdorff space Y, then $\{x \in X : f(x) = g(x)\}$ is closed in X; in particular, if f and g agree on a dense subset of X, then $f = g$ on X.

9. Let X be a topological space. Show that the set $C(X) = \{f \colon X \to \mathbb{R} : f \text{ is continuous}\}$ is a linear subspace of the real vector space of all functions $X \to \mathbb{R}$. Show that if $f_n \to f$ locally uniformly on X and $f_n \in C(X)$ for all n, then $f \in C(X)$. Deduce that $C_b(X) = C(X) \cap \ell_\infty(X)$ is complete in the uniform metric.

10. (a) Show that a space with a countable base is separable and that every separable *metric* space has a countable base. Deduce that a subspace of a separable metric space is separable.

(b) Prove that the family of all half-open intervals $[a, b)$ in $\mathbb{R}$ is a base for a topology τ on $\mathbb{R}$. Let $X = (\mathbb{R}, \tau)$. Show that X is separable but has no countable base. Show that $X \times X$ with the product topology is separable. Identify the subspace topology on $Y = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. Is Y separable?

11. (a) Show that there is a metric on $\mathbb{R}$ which is equivalent to the usual metric but in which $\mathbb{R}$ is not complete.

(b) Let d and d' be equivalent metrics on a set M. Show that if d and d' are uniformly equivalent, then (M, d) and (M, d') have the same Cauchy sequences (and hence one is complete if and only if the other is complete). If (M, d) and (M, d') have the same Cauchy sequences, does it follow that d and d' are uniformly equivalent?

12. Let f be a contraction mapping on a non-empty complete metric space M (so Lipschitz with constant $\lambda < 1$). Fix $x_0 \in M$ and define (x_n) recursively by $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. The proof of the Contraction Mapping Theorem shows that (x_n) converges to the unique fixed point z of f. Prove that $d(x_n, z) \leqslant \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

Show that $\cos x = x$ has a unique solution in $\mathbb{R}$. Using a calculator, find a good approximation to this solution and justify the claimed accuracy of your approximation using the above result.

13. Let $f \colon M \to M$ be a function on a non-empty complete metric space M. Assume that for some $k \geqslant 1$, the k-fold composition $f \circ \cdots \circ f$ of f with itself is a contraction mapping. Show that f has a unique fixed point. Deduce that for any $y \in \mathbb{R}$, the initial value problem

$$f'(t) = f(t^2) \ , \qquad f(0) = y$$

has a unique solution on the interval $[0, 1]$.

14. We are given a nested sequence $A_1 \supset A_2 \supset \dots$ of non-empty closed subsets of a complete metric space. Assume that the diameter $\operatorname{diam}(A_n) = \sup\{d(x, y) : x, y \in A_n\}$ converges to zero. Show that the intersection $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty. Is it true that a nested sequence of closed balls in a complete metric space has non-empty intersection?