ANALYSIS II EXAMPLES 4

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. For each of the following sets X, determine whether the given function d defines a metric on X:

(i) $X = \mathbb{R}^n$, $d(x,y) = \min\{|x_1 - y_1|, \dots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$, $d(x,x) = 0$ for all x, otherwise $d(x,y) = 2^n$ if $x - y = 2^n a$ where a is odd.

(iii) $X = \mathbb{Q}$, $d(x,x) = 0$ for all x, otherwise $d(x,y) = 3^{-n}$ if $x - y = 3^n a/b$ where a, b are prime to 3 (and n may be positive, negative or zero).

(iv) $X = \{\text{functions } \mathbb{N} \to \mathbb{N}\}$, $d(f,f) = 0$, otherwise $d(f,g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(v) $X = \mathbb{C}$, $d(z,w) = |z - w|$ if z and w are on the same straight line through 0, otherwise $d(z,w) = |z| + |w|$.

2. A metric d on a set X is called an *ultrametric* if it satisfies the following stronger form of the triangle inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x, y, z \in X\ .$$

Which of the metrics in question 1 are ultrametrics? Show that in an ultrametric space 'every triangle is isosceles' (that is, at least two of $d(x,y)$, $d(y,z)$ and $d(x,z)$ must be equal), and deduce that every open ball in an ultrametric space is a closed set. Does it follow that every open set must be closed?

3. There is a persistent 'urban myth' about the mathematics research student who spent three years writing a thesis about properties of 'antimetric spaces', where an *antimetric* on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the same axioms as a metric except that the triangle inequality is reversed (i.e. $d(x,z) \geq d(x,y) + d(y,z)$ for all x, y, z). Why would such a thesis be unlikely to be considered worth a Ph.D.?

***4**. [Tripos IB 93301(b)] Let (X,d) be a metric space without isolated points (i.e. such that $\{x\}$ is not open for any $x \in X$), and $(x_n)_{n\geq 0}$ a sequence of points of X. Show that it is possible to find a sequence of points y_n of X and positive real numbers r_n such that $r_n \to 0$, $d(x_n, y_n) > r_n$ and

$$B(y_n, r_n) \subseteq B(y_{n-1}, r_{n-1})$$

for each $n > 0$. Deduce that a nonempty complete metric space without isolated points has uncountably many points. [This is a direct generalization of the familiar proof of uncountability of $\mathbb{R}$ using decimal expansions: can you see why?]

5. (i) Consider the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ such that all but finitely many of the a_n are zero, introduced in Sheet 2, Exercise 11. Show that the norm defined by

$$||\mathbf{a}||_1 = \sum_{n=1}^{\infty} |a_n|$$

is not complete.

(ii) Consider $\mathcal{C}[a,b]$ the space of continuous functions on $[a,b]$ and show that

$$d(f,g) = \int_a^b |f(x) - g(x)|\, dx,$$

is a metric. Is $(\mathcal{C}[a,b], d)$ complete?

6. [Tripos IB 96401(b), modified] (i) Let (X,d) be a nonempty complete metric space, and let $f : X \to X$ be a continuous map such that, for any two points x, y of X, the sum $\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$ converges. Show that f has a unique fixed point.

(ii) By considering the function $x \mapsto \max\{x-1, 0\}$ on the interval $[0,\infty) \subseteq \mathbb{R}$, show that a function satisfying the hypotheses of (i) need not be a contraction mapping.

(iii) Let ϕ be a continuous real-valued function on $\mathbb{R} \times [a,b]$ which satisfies the Lipschitz condition

$$|\phi(x,t) - \phi(y,t)| \leq M\,|x-y| \ , \text{ for all } x, y \in \mathbb{R} \text{ and } t \in [a,b],$$

and let $g \in \mathcal{C}[a,b]$. Define $F\colon \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ by

$$F(h)(t) = g(t) + \int_a^t \phi(h(s), s)\, ds \ .$$

Show by induction that

$$|F^n(h)(t) - F^n(k)(t)| \leq \frac{1}{n!} M^n (t-a)^n \, \|h-k\|_\infty \ ,$$

for all $h, k \in \mathcal{C}[a,b]$ and $a \leq t \leq b$, and deduce that F has a unique fixed point.

(iv) In the original 1996 Tripos question from which this question was adapted, the word 'continuous' in the second line of part (i) was accidentally omitted. Give a counterexample to the result which the 1996 IB students were asked to prove.

7. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that if (X, d_X) is compact then any continuous function $f : X \to Y$ is uniformly continuous.

8. [Tripos IB 95401(b)] For which a and b, with $a \leq 0 \leq b$, is the mapping $T : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by

$$T(f)(x) = 1 + \int_0^x 2t\, f(t)\, dt$$

a contraction? Deduce that the differential equation

$$\frac{dy}{dx} = 2xy \ , \quad \text{with } y = 1 \text{ when } x = 0 \ ,$$

has a unique solution in some interval containing 0. In what interval can the differential equation be solved?

9. A mapping $f(X,d) \to (Y,d')$ between metric spaces is called an *isometric embedding* if it preserves distances exactly, i.e. $d'(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.

(i) Show that an isometric embedding is necessarily injective.

(ii) Suppose (X,d) is compact and let $f : (X,d) \to (X,d)$ be an isometric embedding. Show that f is surjective. [Method: suppose x is not in the image of f, and derive a contradiction by considering the distances between terms of the sequence $(x, f(x), f(f(x)), \dots)$.]

(iii) Give an example to show that compactness cannot be weakened to completeness in (ii).

(iv) Let (X,d) be a bounded metric space, and let V be the vector space of bounded continuous real-valued functions on X, equipped with the uniform norm (i.e. $\|f\| = \sup\{|f(x)| : x \in X\}$). Show that there is an isometric embedding $X \to V$. [Thus, up to isometry, every bounded metric space is a subspace of a normed space.]

*__10__. Let (X,d) be a metric space, and let $\mathcal{H}X$ denote the set of nonempty closed bounded subsets of X. ($\mathcal{H}X$ is sometimes called the *hyperspace* of X.)

(i) For $x \in X$ and $F \in \mathcal{H}X$, we define the distance from x to F to be

$$\overline{d}(x, F) = \inf\{d(x, y) : y \in F\} \ .$$

Show that $\overline{d}(x, F) = 0$ if and only if $x \in F$, and that $\overline{d}(x, F) \leq d(x, y) + \overline{d}(y, F)$ for any x, y and F. [Warning: the infimum in the definition of $\overline{d}$ need **not** be attained if $x \notin F$.]

(ii) Now we define the distance between two elements of $\mathcal{H}X$ by the formula

$$\overline{\overline{d}}(F, G) = \sup(\{\overline{d}(x, G) : x \in F\} \cup \{\overline{d}(y, F) : y \in G\}) \ .$$

Verify that $\overline{\overline{d}}$ is a metric on $\mathcal{H}X$.

(iii) Show that the function which sends x to $\{x\}$ is an isometric embedding $X \to \mathcal{H}X$. Show also that its image is a closed subset of $\mathcal{H}X$.

(iv) Show that the function $(F, G) \mapsto F \cup G$ is a continuous mapping $\mathcal{H}X \times \mathcal{H}X \to \mathcal{H}X$. Is $(F, G) \mapsto F \cap G$ continuous?

(v) Show that $\mathcal{H}X$ is complete if and only if X is complete. [One direction follows from (iii); for the other, suppose given a Cauchy sequence (F_n) in $\mathcal{H}X$, and consider the set of all limits in X of sequences (x_n) such that $x_n \in F_n$ for all n. It is helpful to begin by showing that this set coincides with $\{x \in X : \overline{d}(x, F_n) \to 0 \text{ as } n \to \infty\}$.]

(vi) Show that $\mathcal{H}X$ is compact if and only if X is compact. [Use the fact that compactness is equivalent to 'complete and totally bounded': if $X_0 = \{x_1, x_2, \ldots, x_n\}$ is a finite set of points in X such that the balls $B(x_i, \epsilon)$ cover X, consider the $2^n - 1$ points of $\mathcal{H}X$ which are the nonempty subsets of X_0.]

(vii) Suppose X is compact, and that $f : X \to X$ is a contraction mapping. Show that the function $\overline{f}$ defined by $\overline{f}(F) = \{f(x) : x \in F\}$ maps $\mathcal{H}X$ into itself, and that it is a contraction mapping. What is its unique fixed point?