## ANALYSIS II EXAMPLES 4

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The questions on this sheet are not all equally difficult and the harder ones are marked with \*'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at g.p.paternain@dpmms.cam.ac.uk. The questions are based on the example sheets I gave last year, but I have made a few changes.

**1.** For each of the following sets X, determine whether the given function d defines a metric on X: (i)  $X = \mathbb{R}^n$ ,  $d(x, y) = \min\{|x_1 - y_1|, \dots, |x_n - y_n|\}$ .

(ii)  $X = \mathbb{Z}$ , d(x, x) = 0 for all x, otherwise  $d(x, y) = 2^n$  if  $x - y = 2^n a$  where a is odd.

(iii)  $X = \mathbb{Q}$ , d(x, x) = 0 for all x, otherwise  $d(x, y) = 3^{-n}$  if  $x - y = 3^n a/b$  where a, b are prime to 3 (and n may be positive, negative or zero).

(iv)  $X = \{ \text{functions } \mathbb{N} \to \mathbb{N} \}, d(f, f) = 0$ , otherwise  $d(f, g) = 2^{-n}$  for the least n such that  $f(n) \neq g(n)$ .

(v)  $X = \mathbb{C}$ , d(z, w) = |z - w| if z and w are on the same straight line through 0, otherwise d(z, w) = |z| + |w|.

**2**. A metric d on a set X is called an *ultrametric* if it satisfies the following stronger form of the triangle inequality:

 $d(x,z) \le \max\{d(x,y), d(y,z)\} \quad \text{for all } x, y, z \in X .$ 

Which of the metrics in question 1 are ultrametrics? Show that in an ultrametric space 'every triangle is isosceles' (that is, at least two of d(x, y), d(y, z) and d(x, z) must be equal), and deduce that every open ball in an ultrametric space is a closed set. Does it follow that every open set must be closed?

**3.** There is a persistent 'urban myth' about the mathematics research student who spent three years writing a thesis about properties of 'antimetric spaces', where an *antimetric* on a set X is a function  $d: X \times X \to \mathbb{R}$  satisfying the same axioms as a metric except that the triangle inequality is reversed (i.e.  $d(x, z) \ge d(x, y) + d(y, z)$  for all x, y, z). Why would such a thesis be unlikely to be considered worth a Ph.D.?

\*4. [Tripos IB 93301(b)] Let (X, d) be a metric space without isolated points (i.e. such that  $\{x\}$  is not open for any  $x \in X$ ), and  $(x_n)_{n\geq 0}$  a sequence of points of X. Show that it is possible to find a sequence of points  $y_n$  of X and positive real numbers  $r_n$  such that  $r_n \to 0$ ,  $d(x_n, y_n) > r_n$  and

$$B(y_n, r_n) \subseteq B(y_{n-1}, r_{n-1})$$

for each n > 0. Deduce that a nonempty complete metric space without isolated points has uncountably many points. [This is a direct generalization of the familiar proof of uncountability of  $\mathbb{R}$ using decimal expansions: can you see why?]

5. (i) Consider the space of real sequences  $\mathbf{a} = (a_n)_{n=1}^{\infty}$  such that all but finitely many of the  $a_n$  are zero, introduced in Sheet 2, Exercise 11. Show that the norm defined by

$$||\mathbf{a}||_1 = \sum_{n=1}^{\infty} |a_n|$$

is not complete.

(ii) Consider  $\mathcal{C}[a, b]$  the space of continuous functions on [a, b] and show that

$$d(f,g) = \int_a^b |f(x) - g(x)| \, dx,$$

is a metric. Is  $(\mathcal{C}[a, b], d)$  complete?

**6.** [Tripos IB 96401(b), modified] (i) Let (X, d) be a nonempty complete metric space, and let  $f: X \to X$  be a continuous map such that, for any two points x, y of X, the sum  $\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$  converges. Show that f has a unique fixed point.

(ii) By considering the function  $x \mapsto \max\{x-1,0\}$  on the interval  $[0,\infty) \subseteq \mathbb{R}$ , show that a function satisfying the hypotheses of (i) need not be a contraction mapping.

(iii) Let  $\phi$  be a continuous real-valued function on  $\mathbb{R} \times [a, b]$  which satisfies the Lipschitz condition

$$|\phi(x,t) - \phi(y,t)| \le M |x-y|$$
, for all  $x, y \in \mathbb{R}$  and  $t \in [a,b]$ ,

and let  $g \in \mathcal{C}[a, b]$ . Define  $F : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$  by

$$F(h)(t) = g(t) + \int_a^t \phi(h(s), s) \, ds \; .$$

Show by induction that

$$|F^{n}(h)(t) - F^{n}(k)(t)| \le \frac{1}{n!}M^{n}(t-a)^{n} ||h-k||_{\infty}$$

for all  $h, k \in \mathcal{C}[a, b]$  and  $a \leq t \leq b$ , and deduce that F has a unique fixed point.

(iv) In the original 1996 Tripos question from which this question was adapted, the word 'continuous' in the second line of part (i) was accidentally omitted. Give a counterexample to the result which the 1996 IB students were asked to prove.

7. Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Show that if  $(X, d_X)$  is compact then any continuous function  $f: X \to Y$  is uniformly continuous.

8. [Tripos IB 95401(b)] For which a and b, with  $a \leq 0 \leq b$ , is the mapping  $T : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$  defined by

$$T(f)(x) = 1 + \int_0^x 2t f(t) dt$$

a contraction? Deduce that the differential equation

$$\frac{dy}{dx} = 2xy$$
, with  $y = 1$  when  $x = 0$ ,

has a unique solution in some interval containing 0. In what interval can the differential equation be solved?

**9**. A mapping  $f(X, d) \to (Y, d')$  between metric spaces is called an *isometric embedding* if it preserves distances exactly, i.e. d'(f(x), f(y)) = d(x, y) for all  $x, y \in X$ .

(i) Show that an isometric embedding is necessarily injective.

(ii) Suppose (X, d) is compact and let  $f: (X, d) \to (X, d)$  be an isometric embedding. Show that f is surjective. [Method: suppose x is not in the image of f, and derive a contradiction by considering the distances between terms of the sequence  $(x, f(x), f(f(x)), \ldots)$ .]

(iii) Give an example to show that compactness cannot be weakened to completeness in (ii).

(iv) Let (X, d) be a bounded metric space, and let V be the vector space of bounded continuous real-valued functions on X, equipped with the uniform norm (i.e.  $||f|| = \sup\{|f(x)| : x \in X\}$ ). Show that there is an isometric embedding  $X \to V$ . [Thus, up to isometry, every bounded metric space is a subspace of a normed space.]

\*10. Let (X, d) be a metric space, and let  $\mathcal{H}X$  denote the set of nonempty closed bounded subsets of X. ( $\mathcal{H}X$  is sometimes called the *hyperspace* of X.)

(i) For  $x \in X$  and  $F \in \mathcal{H}X$ , we define the distance from x to F to be

$$\overline{d}(x,F) = \inf\{d(x,y) : y \in F\}.$$

Show that  $\overline{d}(x,F) = 0$  if and only if  $x \in F$ , and that  $\overline{d}(x,F) \leq d(x,y) + \overline{d}(y,F)$  for any x, y and F. [Warning: the infimum in the definition of  $\overline{d}$  need **not** be attained if  $x \notin F$ .]

(ii) Now we define the distance between two elements of  $\mathcal{H}X$  by the formula

$$\overline{d}(F,G) = \sup(\{\overline{d}(x,G) : x \in F\} \cup \{\overline{d}(y,F) : y \in G\}).$$

Verify that  $\overline{d}$  is a metric on  $\mathcal{H}X$ .

(iii) Show that the function which sends x to  $\{x\}$  is an isometric embedding  $X \to \mathcal{H}X$ . Show also that its image is a closed subset of  $\mathcal{H}X$ .

(iv) Show that the function  $(F,G) \mapsto F \cup G$  is a continuous mapping  $\mathcal{H}X \times \mathcal{H}X \to \mathcal{H}X$ . Is  $(F,G) \mapsto F \cap G$  continuous?

(v) Show that  $\mathcal{H}X$  is complete if and only if X is complete. [One direction follows from (iii); for the other, suppose given a Cauchy sequence  $(F_n)$  in  $\mathcal{H}X$ , and consider the set of all limits in X of sequences  $(x_n)$  such that  $x_n \in F_n$  for all n. It is helpful to begin by showing that this set coincides with  $\{x \in X : \overline{d}(x, F_n) \to 0 \text{ as } n \to \infty\}$ .]

(vi) Show that  $\mathcal{H}X$  is compact if and only if X is compact. [Use the fact that compactness is equivalent to 'complete and totally bounded': if  $X_0 = \{x_1, x_2, \ldots, x_n\}$  is a finite set of points in X such that the balls  $B(x_i, \epsilon)$  cover X, consider the  $2^n - 1$  points of  $\mathcal{H}X$  which are the nonempty subsets of  $X_0$ .]

(vii) Suppose X is compact, and that  $f: X \to X$  is a contraction mapping. Show that the function  $\overline{f}$  defined by  $\overline{f}(F) = \{f(x) : x \in F\}$  maps  $\mathcal{H}X$  into itself, and that it is a contraction mapping. What is its unique fixed point?